Third order solutions of the cosmological density perturbations in the Horndeski's most general scalar-tensor theory with the Vainshtein mechanism

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We study the third order solutions of the cosmological density perturbations in the Horndeski's most general scalar-tensor theory under the condition that the Vainshtein mechanism is at work. In this work, we thoroughly investigate the independence property of the functions describing the nonlinear mode-couplings, which is also useful for models within the general relativity. Then, we find that the solutions of the density contrast and the velocity divergence up to the third order ones are characterized by 6 parameters. Furthermore, the 1-loop order power spectra obtained with the third order solutions are described by 4 parameters. We exemplify the behavior of the 1-loop order power spectra assuming the kinetic gravity braiding model, which demonstrates that the effect of the modified gravity appears more significantly in the power spectrum of the velocity divergence than the density contrast.

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I. INTRODUCTION

The accelerated expansion of the Universe is one of the most fundamental problems in modern cosmology. The standard cosmological model introducing the cosmological constant is consistent with various observations [1, 2]. However, the small value of the cosmological constant raises the problem of fine tuning [3–5]. As an alternative to the cosmological constant, the cosmic accelerated expansion might be explained by modifying gravity theory, e.g., [6–18]. In the present paper, we focus on the most general scalar-tensor theory with the second order differential field equations [19, 20], which was first discovered by Horndeski [21]. The Horndeski's most general scalar-tensor theory, including 4 arbitrary functions of the scalar field and kinetic term, reduces to various modified gravity models by choosing the specific 4 functions. Because the Horndeski's theory includes a wide class of modified gravity models, we adopt it as an effective theory of the generalized theories of gravity.

In the present paper, we investigate the aspects of the quasi-nonlinear evolution of the cosmological density perturbations in the Horndeski's most general scalar-tensor theory assuming that the Vainshtein mechanism is at work [22–25]. The Vainshtein mechanism is the screening mechanisms, which is useful to evade the constraints from the gravity tests in the solar system. We investigate the effects of the nonlinear terms in the matter's fluid equations as well as the nonlinear derivative interaction terms in the scalar field equation. In a previous work [26], the second order solution of the cosmological density perturbations is obtained. In the present paper, we extend the analysis to the third order solution, which enables us to compute the 1-loop order matter power spectrum.

There are many works on the higher order cosmological density perturbations and the quasi-nonlinear matter power spectrum, which have been developed from the standard perturbative approach (see e.g., [27–37]). Improvements to include the non-perturbative effects have been investigated, e.g., [38–43], but we here adopt the standard perturbative approach of the cosmological density perturbations as a starting place for the analysis of the Horndeski's most general scalar-tensor theory. Related to the work of the

present paper, we refer the recent work by Lee, Park and Biern [44], in which a similar solution is obtained for the dark energy model within the general relativity.

This paper is organized as follows. In section 2, we review the basic equations and the second order solution [26]. In section 3, we construct the third order solutions of the cosmological density perturbations. Here, we carefully investigate independent functions of mode-couplings describing nonlinear interactions. In section 4, we derive the expression of the 1-loop order power spectra of the matter density contrast and the velocity divergence. In section 5, an expression for the trispectrum for the density contrast is presented. In section 6, we demonstrate the behavior of the 1-loop order power spectra in the kinetic gravity braiding model. Section 7 is devoted to summary and conclusions. In appendix A, definitions of the coefficients to characterize the Horndeski's theory are summarized. In appendix B, definitions of the functions to describe the nonlinear mode-coupling for the third order solutions are summarized. In appendix C, a derivation of the 1-loop power spectra is summarized. Expressions in appendix D are useful for the deviation of the 1-loop power spectra. Appendix E lists the coefficients to characterize the kinetic gravity braiding model.

II. REVIEW OF THE SECOND ORDER SOLUTION

Let us start with reviewing the basic formulas [23, 26]. We consider the Horndeski's most general scalar-tensor theory, whose action is given by

$$S=\int d^4x\sqrt{-g}\left(\mathcal{L}_{\rm GG}+\mathcal{L}_{\rm m}\right), \tag{1}$$

where we define

$$\begin{aligned}\mathcal{L}_{\rm GG} =& K(\phi,X)-G_3(\phi,X)\Box\phi+G_4(\phi,X)R+G_{4X}\left[(\Box\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right]\\ &+G_5(\phi,X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi-\frac{1}{6}G_{5X}\left[(\Box\phi)^3-3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2+2(\nabla_\mu\nabla_\nu\phi)^3\right],\end{aligned} \tag{2}$$

where K, G_3, G_4, and G_5 are arbitrary function of the scalar field ϕ and the kinetic term $X:=-(\partial\phi)^2/2$, G_{iX} denotes $\partial G_i/\partial X$, R is the Ricci scalar, $G_{\mu\nu}$ is the Einstein tensor, and $\mathcal{L}_{\rm m}$ is the Lagrangian of the matter field, which is minimally coupled to the gravity.

The basic equations for the cosmological density perturbations are derived in Ref. [23]. Here, we briefly review the method and the results (see [23] for details). This theory is discovered in [19] as a generalization of the galileon theory ([45], see also [20, 46–64]), but the equivalence with the Horndeski's theory [21] is shown in [20]. We consider a spatially flat expanding universe and the metric perturbations in the Newtonian gauge, whose line element is written as

$$ds^2=-(1+2\Phi(t,\mathbf{x}))dt^2+a^2(t)(1-2\Psi(t,\mathbf{x}))d\mathbf{x}^2. \tag{3}$$

We define the scalar field with perturbations by

$$\phi\,\rightarrow\,\phi(t)+\delta\phi(t,\mathbf{x}), \tag{4}$$

and we introduce $Q=H\delta\phi/\dot{\phi}$.

The basic equations of the gravitational and scalar fields are derived on the basis of the quasi-static approximation of the subhorizon scales [23]. In the models that the Vainshtein mechanism may work, the basic equations can be found by keeping the leading terms schematically written as $(\partial\partial Y)^n$, with $n\geq 1$, where ∂ denotes a spatial derivative and Y does any of Φ, Ψ or Q. Such terms make a leading contribution of the order $(L_{\rm H}^2\partial\partial Y)^n$, where $L_{\rm H}$ is a typical horizon length scale, and we have

$$\nabla^2\left(\mathcal{F}_T\Psi-\mathcal{G}_T\Phi-A_1Q\right)=\frac{B_1}{2a^2H^2}\mathcal{Q}^{(2)}+\frac{B_3}{a^2H^2}\left(\nabla^2\Phi\nabla^2Q-\partial_i\partial_j\Phi\partial^i\partial^jQ\right), \tag{5}$$

and

$$\mathcal{G}_T\nabla^2\Psi = \frac{a^2}{2}\rho_{\rm m}\delta - A_2\nabla^2 Q - \frac{B_2}{2a^2H^2}\mathcal{Q}^{(2)} - \frac{B_3}{a^2H^2}\left(\nabla^2\Psi\nabla^2 Q - \partial_i\partial_j\Psi\partial^i\partial^j Q\right) - \frac{C_1}{3a^4H^4}\mathcal{Q}^{(3)}, \quad (6)$$

where $\rho_{\rm m}$ is the background matter density and δ is the matter density contrast, and we define

$$\mathcal{Q}^{(2)} := \left(\nabla^2 Q\right)^2 - \left(\partial_i\partial_j Q\right)^2, \quad (7)$$

$$\mathcal{Q}^{(3)} := \left(\nabla^2 Q\right)^3 - 3\nabla^2 Q\left(\partial_i\partial_j Q\right)^2 + 2\left(\partial_i\partial_j Q\right)^3. \quad (8)$$

The equation of the scalar field perturbation is

$$A_0\nabla^2 Q - A_1\nabla^2\Psi - A_2\nabla^2\Phi + \frac{B_0}{a^2H^2}\mathcal{Q}^{(2)} - \frac{B_1}{a^2H^2}\left(\nabla^2\Psi\nabla^2 Q - \partial_i\partial_j\Psi\partial^i\partial^j Q\right) - \frac{B_2}{a^2H^2}\left(\nabla^2\Phi\nabla^2 Q - \partial_i\partial_j\Phi\partial^i\partial^j Q\right) - \frac{B_3}{a^2H^2}\left(\nabla^2\Phi\nabla^2\Psi - \partial_i\partial_j\Phi\partial^i\partial^j\Psi\right) - \frac{C_0}{a^4H^4}\mathcal{Q}^{(3)} - \frac{C_1}{a^4H^4}\mathcal{U}^{(3)} = 0, \quad (9)$$

where we define

$$\mathcal{U}^{(3)} := \mathcal{Q}^{(2)}\nabla^2\Phi - 2\nabla^2 Q\partial_i\partial_j Q\partial^i\partial^j\Phi + 2\partial_i\partial_j Q\partial^j\partial^k Q\partial_k\partial^i\Phi. \quad (10)$$

Here the coefficients $\mathcal{F}_T$, A_1, B_1, C_1, etc., are defined in Appendix A. A_i, B_i, and C_i are the coefficients of the linear, quadratic and cubic terms of Ψ, Φ, and Q, respectively.

From the continuity equation and the Euler equation for the matter fluid, we have the following equations for the density contrast δ and the velocity field u^i,

$$\frac{\partial\delta(t,\mathbf{x})}{\partial t} + \frac{1}{a}\partial_i[(1+\delta(t,\mathbf{x}))u^i(t,\mathbf{x})] = 0, \quad (11)$$

$$\frac{\partial u^i(t,\mathbf{x})}{\partial t} + \frac{\dot a}{a}u^i(t,\mathbf{x}) + \frac{1}{a}u^j(t,\mathbf{x})\partial_j u^i(t,\mathbf{x}) = -\frac{1}{a}\partial_i\Phi(t,\mathbf{x}), \quad (12)$$

respectively. The properties of the gravity sector is influenced through Φ in (12), where Φ is determined by Eqs. (5), (6) and (9).

Now introducing the scalar function $\theta \equiv \nabla\mathbf{u}/(aH)$, which we call velocity divergence, and we perform the Fourier expansions for δ and θ,

$$\delta(t,\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3p\,\delta(t,\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (13)$$

$$u^j(t,\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3p\frac{-ip^j}{p^2}aH\theta(t,\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}. \quad (14)$$

In the similar way, we perform the Fourier expansions for Φ, Ψ, and Q. Then, the equations for the gravity (5) and (6) lead to

$$-p^2\left(\mathcal{F}_T\Psi(t,\mathbf{p}) - \mathcal{G}_T\Phi(t,\mathbf{p}) - A_1 Q(t,\mathbf{p})\right) = \frac{B_1}{2a^2H^2}\Gamma[t,\mathbf{p};Q,Q] + \frac{B_3}{a^2H^2}\Gamma[t,\mathbf{p};Q,\Phi], \quad (15)$$

$$-p^2(\mathcal{G}_T\Psi(t,\mathbf{p}) + A_2 Q(t,\mathbf{p})) - \frac{a^2}{2}\rho_{\rm m}\delta(t,\mathbf{p}) = -\frac{B_2}{2a^2H^2}\Gamma[t,\mathbf{p};Q,Q] - \frac{B_3}{a^2H^2}\Gamma[t,\mathbf{p};Q,\Psi] - \frac{C_1}{3a^4H^4}\Xi_1[t,\mathbf{p};Q,Q,Q], \quad (16)$$

respectively, where we define

$$\Gamma[t,\mathbf{p};Z_1,Z_2]=\frac{1}{(2\pi)^3}\int d\mathbf{k}_1 d\mathbf{k}_2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})\left(k_1^2k_2^2-(\mathbf{k}_1\cdot\mathbf{k}_2)^2\right)Z_1(t,\mathbf{k}_1)Z_2(t,\mathbf{k}_2), \tag{17}$$

$$\Xi_1[t,\mathbf{p};Z_1,Z_2,Z_3]=\frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})$$
$$\times\left[-k_1^2k_2^2k_3^2+3k_1^2(\mathbf{k}_2\cdot\mathbf{k}_3)^2-2(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{k}_3)(\mathbf{k}_3\cdot\mathbf{k}_1)\right]Z_1(t,\mathbf{k}_1)Z_2(t,\mathbf{k}_2)Z_3(t,\mathbf{k}_3), \tag{18}$$

where Z_1, Z_2 and Z_3 denote any of Q, Φ or Ψ. The equation for scalar field perturbation (9) leads to

$$\begin{aligned}&-p^2(A_0Q(t,\mathbf{p})-A_1\Psi(t,\mathbf{p})-A_2\Phi(t,\mathbf{p}))\\&\quad=-\frac{B_0}{a^2H^2}\Gamma[t,\mathbf{p};Q,Q]+\frac{B_1}{a^2H^2}\Gamma[t,\mathbf{p};Q,\Psi]+\frac{B_2}{a^2H^2}\Gamma[t,\mathbf{p};Q,\Phi]+\frac{B_3}{a^2H^2}\Gamma[t,\mathbf{p};\Psi,\Phi]\\&\qquad+\frac{C_0}{a^4H^4}\Xi_1[t,\mathbf{p};Q,Q,Q]+\frac{C_1}{a^4H^4}\Xi_2[t,\mathbf{p};Q,Q,\Phi],\end{aligned} \tag{19}$$

where we define

$$\begin{aligned}\Xi_2[t,\mathbf{p};Z_1,Z_2,Z_3]=\frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\Big[-k_1^2k_2^2k_3^2+(\mathbf{k}_1\cdot\mathbf{k}_2)^2k_3^2\\+2k_1^2(\mathbf{k}_2\cdot\mathbf{k}_3)^2-2(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{k}_3)(\mathbf{k}_3\cdot\mathbf{k}_1)\Big]Z_1(t,\mathbf{k}_1)Z_2(t,\mathbf{k}_2)Z_3(t,\mathbf{k}_3).\end{aligned} \tag{20}$$

The fluid equations (11) and (12) lead to

$$\frac{1}{H}\frac{\partial\delta(t,\mathbf{p})}{\partial t}+\theta(t,\mathbf{p})=-\frac{1}{(2\pi)^3}\int d\mathbf{k}_1 d\mathbf{k}_2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})\alpha(\mathbf{k}_1,\mathbf{k}_2)\theta(t,\mathbf{k}_1)\delta(t,\mathbf{k}_2), \tag{21}$$

$$\begin{aligned}&\frac{1}{H}\frac{\partial\theta(t,\mathbf{p})}{\partial t}+\left(2+\frac{\dot H}{H^2}\right)\theta(t,\mathbf{p})-\frac{p^2}{a^2H^2}\Phi(t,\mathbf{p})\\&\qquad=-\frac{1}{(2\pi)^3}\int d\mathbf{k}_1 d\mathbf{k}_2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})\beta(\mathbf{k}_1,\mathbf{k}_2)\theta(t,\mathbf{k}_1)\theta(t,\mathbf{k}_2),\end{aligned} \tag{22}$$

where we define

$$\alpha(\mathbf{k}_1,\mathbf{k}_2)=1+\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1^2} \tag{23}$$

$$\beta(\mathbf{k}_1,\mathbf{k}_2)=\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)|\mathbf{k}_1+\mathbf{k}_2|^2}{2k_1^2k_2^2}. \tag{24}$$

Note that $\alpha(\mathbf{k}_1,\mathbf{k}_2)$ doesn't have the symmetry with respect to exchange between $\mathbf{k}_1$ and $\mathbf{k}_2$. We find the solution in terms of a perturbative expansion, which can be written in the form

$$Y(t,\mathbf{p})=\sum_{n=1}Y_n(t,\mathbf{p}), \tag{25}$$

where Y denotes δ, θ, Ψ, Φ or Q, and Y_n denotes the nth order solution of the perturbative expansion.

Neglecting the decaying mode solution, the linear order solution is written as [23, 65]

$$\delta_1(t,\mathbf{p}) \;=\; D_+(t)\delta_{\rm L}(\mathbf{p}), \tag{26}$$
$$\theta_1(t,\mathbf{p}) \;=\; -D_+(t)f(t)\delta_{\rm L}(\mathbf{p}), \tag{27}$$
$$\Phi_1(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}D_+(t)\kappa_\Phi(t)\delta_{\rm L}(\mathbf{p}), \tag{28}$$
$$\Psi_1(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}D_+(t)\kappa_\Psi(t)\delta_{\rm L}(\mathbf{p}), \tag{29}$$
$$Q_1(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}D_+(t)\kappa_Q(t)\delta_{\rm L}(\mathbf{p}), \tag{30}$$

where $D_+(t)$ is growth factor obeying

$$\frac{d^2D_+(t)}{dt^2}+2H\frac{dD_+(t)}{dt}+L(t)D_+(t)=0, \tag{31}$$

with

$$L(t) \;=\; -\frac{(A_0\mathcal{F}_T-A_1^2)\rho_{\rm m}}{2(A_0\mathcal{G}_T^2+2A_1A_2\mathcal{G}_T+A_2\mathcal{F}_T)}, \tag{32}$$

and $\delta_{\rm L}(\mathbf{p})$ describes the linear density perturbations, which are assumed to obey the Gaussian random distribution. Here we adopt the normalization for the growth factor $D_+(a)=a$ at $a\ll 1$, and introduced the linear growth rate defined by $f(t)=d\ln D_+(t)/\ln a$.

The second order solution is written as (see [26] for details),

$$\delta_2(t,\mathbf{p}) \;=\; D_+^2(t)\left(\mathcal{W}_\alpha(\mathbf{p})-\frac{2}{7}\lambda(t)\mathcal{W}_\gamma(\mathbf{p})\right), \tag{33}$$
$$\theta_2(t,\mathbf{p}) \;=\; -D_+^2(t)f\left(\mathcal{W}_\alpha(\mathbf{p})-\frac{4}{7}\lambda_\theta(t)\mathcal{W}_\gamma(\mathbf{p})\right), \tag{34}$$
$$\Phi_2(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}D_+^2(t)(\kappa_\Phi(t)\mathcal{W}_\alpha(\mathbf{p})+\lambda_\Phi(t)\mathcal{W}_\gamma(\mathbf{p})), \tag{35}$$
$$\Psi_2(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}D_+^2(t)(\kappa_\Psi(t)\mathcal{W}_\alpha(\mathbf{p})+\lambda_\Psi(t)\mathcal{W}_\gamma(\mathbf{p})), \tag{36}$$
$$Q_2(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}D_+^2(t)(\kappa_Q(t)\mathcal{W}_\alpha(\mathbf{p})+\lambda_Q(t)\mathcal{W}_\gamma(\mathbf{p})), \tag{37}$$

where the coefficients κ_Φ, κ_Ψ, κ_Q, λ, λ_θ, λ_Φ, λ_Ψ, and λ_Q, are determined by the functions in the Lagrangian and the Hubble parameter, whose definitions are summarized in appendix A. Here $\mathcal{W}_\alpha(\mathbf{p})$ and $\mathcal{W}_\gamma(\mathbf{p})$ are defined as

$$\mathcal{W}_\alpha(\mathbf{p}) \;=\; \frac{1}{(2\pi)^3}\int d\mathbf{k}_1d\mathbf{k}_2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})\alpha^{(s)}(\mathbf{k}_1,\mathbf{k}_2)\delta_{\rm L}(\mathbf{k}_1)\delta_{\rm L}(\mathbf{k}_2), \tag{38}$$
$$\mathcal{W}_\gamma(\mathbf{p}) \;=\; \frac{1}{(2\pi)^3}\int d\mathbf{k}_1d\mathbf{k}_2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})\gamma(\mathbf{k}_1,\mathbf{k}_2)\delta_{\rm L}(\mathbf{k}_1)\delta_{\rm L}(\mathbf{k}_2) \tag{39}$$

with

$$\alpha^{(s)}(\mathbf{k}_1,\mathbf{k}_2) \;=\; 1+\frac{\mathbf{k}_1\cdot\mathbf{k}_2(k_1^2+k_2^2)}{2k_1^2k_2^2}, \tag{40}$$
$$\gamma(\mathbf{k}_1,\mathbf{k}_2) \;=\; 1-\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2k_2^2}, \tag{41}$$

where $\alpha^{(s)}(\mathbf{k}_1,\mathbf{k}_2)$ is obtained by symmetrizing $\alpha(\mathbf{k}_1,\mathbf{k}_2)$ with respect to $\mathbf{k}_1$ and $\mathbf{k}_2$, and $\gamma(\mathbf{k}_1,\mathbf{k}_2)$ is the function to describe the mode-couplings for the nonlinear interaction in the gravitational field equations and the scalar field equation. $\alpha^{(s)}(\mathbf{k}_1,\mathbf{k}_2)$, $\beta(\mathbf{k}_1,\mathbf{k}_2)$ and $\gamma(\mathbf{k}_1,\mathbf{k}_2)$ have the symmetry with respect to exchange between $\mathbf{k}_1$ and $\mathbf{k}_2$. One can easily check that the functions to describe the nonlinear mode-couplings, $\alpha^{(s)}(\mathbf{k}_1,\mathbf{k}_2), \beta(\mathbf{k}_1,\mathbf{k}_2)$, and $\gamma(\mathbf{k}_1,\mathbf{k}_2)$ satisfy

$$\beta(\mathbf{k}_1,\mathbf{k}_2)=\alpha^{(s)}(\mathbf{k}_1,\mathbf{k}_2)-\gamma(\mathbf{k}_1,\mathbf{k}_2). \tag{42}$$

III. THE THIRD ORDER EQUATIONS

In this section we consider the third order solutions. The third order solution of the cosmological density perturbations has been investigated in various models [27–33, 35–37, 44]. We present the third order solution for the Horndeski's theory in the cosmological background. Our results are general and applicable to various modified gravity models. Plus our results are useful for the case of the general relativity because we clarify the independence property of the mode-coupling functions and the relevant parameters to characterize the third order solution. We start with solving the third order equations for gravity and scalar field

$$\begin{aligned}
-p^2\left(\mathcal{F}_T\Psi_3(t,\mathbf{p})-\mathcal{G}_T\Phi_3(t,\mathbf{p})-A_1Q_3(t,\mathbf{p})\right)=\frac{B_1}{a^2H^2}\Gamma[t,\mathbf{p};Q_1,Q_2]+\frac{B_3}{a^2H^2}\Big(\Gamma[t,\mathbf{p};Q_1,\Phi_2]&\\
+\Gamma[t,\mathbf{p};Q_2,\Phi_1]\Big),& \qquad (43)\\
-p^2\left(\mathcal{G}_T\Psi_3(t,\mathbf{p})+A_2Q_3(t,\mathbf{p})\right)-\frac{a^2}{2}\rho_{\rm m}\delta_3(t,\mathbf{p})=-\frac{B_2}{a^2H^2}\Gamma[t,\mathbf{p};Q_1,Q_2]-\frac{B_3}{a^2H^2}\Big(\Gamma[t,\mathbf{p};Q_1,\Psi_2]&\\
+\Gamma[t,\mathbf{p};Q_2,\Psi_1]\Big)-\frac{C_1}{3a^4H^4}\Xi_1[t,\mathbf{p};Q_1,Q_1,Q_1],& \qquad (44)\\
-p^2(A_0Q_3(t,\mathbf{p})-A_1\Psi_3(t,\mathbf{p})-A_2\Phi_3(t,\mathbf{p}))=-\frac{2B_0}{a^2H^2}\Gamma[t,\mathbf{p};Q_1,Q_2]+\frac{B_1}{a^2H^2}\Big(\Gamma[t,\mathbf{p};Q_1,\Psi_2]&\\
+\Gamma[t,\mathbf{p};Q_2,\Psi_1]\Big)+\frac{B_2}{a^2H^2}\Big(\Gamma[t,\mathbf{p};Q_1,\Phi_2]+\Gamma[t,\mathbf{p};Q_2,\Phi_1]\Big)+\frac{B_3}{a^2H^2}\Big(\Gamma[t,\mathbf{p};\Psi_1,\Phi_2]&\\
+\Gamma[t,\mathbf{p};\Psi_2,\Phi_1]\Big)+\frac{C_0}{a^4H^4}\Xi_1[t,\mathbf{p};Q_1,Q_1,Q_1]+\frac{C_1}{a^4H^4}\Xi_2[t,\mathbf{p};Q_1,Q_1,\Phi_1].& \qquad (45)
\end{aligned}$$

Inserting the first and the second order solutions into the above equations, we finally have

$$\begin{aligned}
\mathcal{F}_T\Psi_3(t,\mathbf{p})-\mathcal{G}_T\Phi_3(t,\mathbf{p})-A_1Q_3(t,\mathbf{p})=-\frac{a^2H^2}{p^2}D_+^3(t)\Big(\left(B_1\kappa_Q^2+2B_3\kappa_\Phi\kappa_Q\right)\mathcal{W}_{\gamma\alpha}(\mathbf{p})&\\
+\left(B_1\kappa_Q\lambda_Q+B_3\left(\kappa_\Phi\lambda_Q+\kappa_Q\lambda_\Phi\right)\right)\mathcal{W}_{\gamma\gamma}(\mathbf{p})\Big),& \qquad (46)\\
\mathcal{G}_T\Psi_3(t,\mathbf{p})+A_2Q_3(t,\mathbf{p})+\frac{a^2}{2p^2}\rho_{\rm m}\delta_3(t,\mathbf{p})=\frac{a^2H^2}{p^2}D_+^3(t)\bigg(\left(B_2\kappa_Q^2+2B_3\kappa_\Psi\kappa_Q\right)\mathcal{W}_{\gamma\alpha}(\mathbf{p})&\\
+\left(B_2\kappa_Q\lambda_Q+B_3(\kappa_\Psi\lambda_Q+\kappa_Q\lambda_{\Psi 2})\right)\mathcal{W}_{\gamma\gamma}(\mathbf{p})+\frac{C_1}{3}\kappa_Q^3\mathcal{W}_\xi(\mathbf{p})\bigg),& \qquad (47)\\
A_0Q_3(t,\mathbf{p})-A_1\Psi_3(t,\mathbf{p})-A_2\Phi_3(t,\mathbf{p})=-\frac{a^2H^2}{p^2}D_+^3(t)\bigg(\Big(-2B_0\kappa_Q^2+2B_1\kappa_\Psi\kappa_Q&\\
+2B_2\kappa_\Phi\kappa_Q+2B_3\kappa_\Phi\kappa_\Psi\Big)\mathcal{W}_{\gamma\alpha}(\mathbf{p})+\Big((-2B_0\kappa_Q\lambda_Q+B_1(\kappa_\Psi\lambda_Q+\kappa_Q\lambda_\Psi)&\\
+B_2(\kappa_\Phi\lambda_Q+\kappa_Q\lambda_\Phi)+B_3(\kappa_\Phi\lambda_\Psi+\kappa_\Psi\lambda_\Phi))\mathcal{W}_{\gamma\gamma}(\mathbf{p})+(C_0\kappa_Q^3+C_1\kappa_\Phi\kappa_Q^2)\mathcal{W}_\xi(\mathbf{p})\bigg),& \qquad (48)
\end{aligned}$$

where we define $\mathcal{W}_{\gamma\alpha}(\mathbf{p})$, $\mathcal{W}_{\gamma\gamma}(\mathbf{p})$ and $\mathcal{W}_{\xi}(\mathbf{p})$ by Eqs. (B1), (B2) and (B3), respectively, in appendix B. Then, the gravitational and the curvature potentials, and the scalar field perturbations are written as

$$\Phi_3(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}\left(\kappa_\Phi(t)\delta_3(t,\mathbf{p}) + D_+^3(t)\left(\sigma_\Phi(t)\mathcal{W}_{\gamma\alpha}(\mathbf{p}) + \mu_\Phi(t)\mathcal{W}_{\gamma\gamma}(\mathbf{p}) + \nu_\Phi(t)\mathcal{W}_\xi(\mathbf{p})\right)\right), \qquad (49)$$

$$\Psi_3(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}\left(\kappa_\Psi(t)\delta_3(t,\mathbf{p}) + D_+^3(t)\left(\sigma_\Psi(t)\mathcal{W}_{\gamma\alpha}(\mathbf{p}) + \mu_\Psi(t)\mathcal{W}_{\gamma\gamma}(\mathbf{p}) + \nu_\Psi(t)\mathcal{W}_\xi(\mathbf{p})\right)\right), \qquad (50)$$

$$Q_3(t,\mathbf{p}) \;=\; -\frac{a^2H^2}{p^2}\left(\kappa_Q(t)\delta_3(t,\mathbf{p}) + D_+^3(t)\left(\sigma_Q(t)\mathcal{W}_{\gamma\alpha}(\mathbf{p}) + \mu_Q(t)\mathcal{W}_{\gamma\gamma}(\mathbf{p}) + \nu_Q(t)\mathcal{W}_\xi(\mathbf{p})\right)\right), \qquad (51)$$

where the coefficients $\sigma_\Phi(t)$ $\mu_\Phi(t)$, $\nu_\Phi(t)$, etc., are defined in appendix A. The third order equations for $\delta_3(t,\mathbf{p})$ and $\theta_3(t,\mathbf{p})$ are,

$$\begin{aligned}&\frac{1}{H}\frac{\partial\delta_3(t,\mathbf{p})}{\partial t} + \theta_3(t,\mathbf{p})\\ &\qquad = -\frac{1}{(2\pi)^3}\int d\mathbf{k}_1 d\mathbf{k}_2 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})\alpha(\mathbf{k}_1,\mathbf{k}_2)(\theta_1(t,\mathbf{k}_1)\delta_2(t,\mathbf{k}_2)+\theta_2(t,\mathbf{k}_1)\delta_1(t,\mathbf{k}_2)), \qquad (52)\\ &\frac{1}{H}\frac{\partial\theta_3(t,\mathbf{p})}{\partial t} + \left(2+\frac{\dot H}{H^2}\right)\theta_3(t,\mathbf{p}) - \frac{p^2}{a^2H^2}\Phi_3(t,\mathbf{p})\\ &\qquad = -\frac{2}{(2\pi)^3}\int d\mathbf{k}_1 d\mathbf{k}_2 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{p})\beta(\mathbf{k}_1,\mathbf{k}_2)\theta_1(t,\mathbf{k}_1)\theta_2(t,\mathbf{k}_2). \qquad (53)\end{aligned}$$

Using the first and the second order solutions, these equations are rewritten as

$$\begin{aligned}&\frac{1}{H}\frac{\partial\delta_3(t,\mathbf{p})}{\partial t} + \theta_3(t,\mathbf{p})\\ &\qquad = D_+^3(t) f\left(\mathcal{W}_{\alpha\alpha R}(\mathbf{p}) - \frac{2}{7}\lambda\mathcal{W}_{\alpha\gamma R}(\mathbf{p}) + \mathcal{W}_{\alpha\alpha L}(\mathbf{p}) - \frac{4}{7}\lambda_\theta\mathcal{W}_{\alpha\gamma L}(\mathbf{p})\right), \qquad (54)\\ &\frac{1}{H}\frac{\partial\theta_3(t,\mathbf{p})}{\partial t} + \left(2+\frac{\dot H}{H^2}\right)\theta_3(t,\mathbf{p}) - \frac{p^2}{a^2H^2}\Phi_3(t,\mathbf{p})\\ &\qquad = 2D_+^3(t) f^2\left(-\mathcal{W}_{\alpha\alpha}(\mathbf{p}) + \frac{4}{7}\lambda_\theta\mathcal{W}_{\alpha\gamma}(\mathbf{p}) + \mathcal{W}_{\gamma\alpha}(\mathbf{p}) - \frac{4}{7}\lambda_\theta\mathcal{W}_{\gamma\gamma}(\mathbf{p})\right), \qquad (55)\end{aligned}$$

where we introduce the functions defined by Eqs. (B7) to (B12), for which we find that the following relations hold,

$$\begin{aligned}2\mathcal{W}_{\gamma\alpha}(\mathbf{p}) &= \mathcal{W}_{\alpha\gamma R}(\mathbf{p}) + 2\mathcal{W}_{\gamma\gamma}(\mathbf{p}) - \mathcal{W}_\xi(\mathbf{p}), \qquad (56)\\ 2\mathcal{W}_{\alpha\alpha}(\mathbf{p}) &= \mathcal{W}_{\alpha\alpha R}(\mathbf{p}) + \mathcal{W}_{\alpha\alpha L}(\mathbf{p}), \qquad (57)\\ 2\mathcal{W}_{\alpha\gamma}(\mathbf{p}) &= \mathcal{W}_{\alpha\gamma R}(\mathbf{p}) + \mathcal{W}_{\alpha\gamma L}(\mathbf{p}). \qquad (58)\end{aligned}$$

Then, Eqs. (54) and (55) reduce to

$$\begin{aligned}&\frac{1}{H}\frac{\partial\delta_3(t,\mathbf{p})}{\partial t} + \theta_3(t,\mathbf{p}) = D_+^3(t) f\left(2\mathcal{W}_{\alpha\alpha}(\mathbf{p}) - \frac{2}{7}\lambda\mathcal{W}_{\alpha\gamma R}(\mathbf{p}) - \frac{4}{7}\lambda_\theta\mathcal{W}_{\alpha\gamma L}(\mathbf{p})\right), \qquad (59)\\ &\frac{1}{H}\frac{\partial\theta_3(t,\mathbf{p})}{\partial t} + \left(2+\frac{\dot H}{H^2}\right)\theta_3(t,\mathbf{p}) - \frac{p^2}{a^2H^2}\Phi_3(t,\mathbf{p}) = 2D_+^3(t) f^2\Big(-\mathcal{W}_{\alpha\alpha}(\mathbf{p})\\ &\qquad + \Big(\frac{2}{7}\lambda_\theta + \frac{1}{2}\Big)\mathcal{W}_{\alpha\gamma R}(\mathbf{p}) + \frac{2}{7}\lambda_\theta\mathcal{W}_{\alpha\gamma L}(\mathbf{p}) + \Big(1 - \frac{4}{7}\lambda_\theta\Big)\mathcal{W}_{\gamma\gamma}(\mathbf{p}) - \frac{1}{2}\mathcal{W}_\xi(\mathbf{p})\Big). \qquad (60)\end{aligned}$$

Combining these two equations, we have the third order equation for $\delta_3(t,\mathbf{p})$ as

$$\frac{\partial^2\delta_3(t,\mathbf{p})}{\partial t^2}+2H\frac{\partial\delta_3(t,\mathbf{p})}{\partial t}+L(t)\delta_3(t,\mathbf{p})=S_{\delta 3}(t,\mathbf{p}), \tag{61}$$

where we define

$$\begin{aligned} S_{\delta 3}(t) \;=\; & D_+^3(t)\Big(N_{\alpha\alpha}(t)\mathcal{W}_{\alpha\alpha}(\mathbf{p})+N_{\alpha\gamma R}(t)\mathcal{W}_{\alpha\gamma R}(\mathbf{p})+N_{\alpha\gamma L}(t)\mathcal{W}_{\alpha\gamma L}(\mathbf{p}) \\ & +N_{\gamma\gamma}(t)\mathcal{W}_{\gamma\gamma}(\mathbf{p})+N_{\xi}(t)\mathcal{W}_{\xi}(\mathbf{p})\Big), \end{aligned} \tag{62}$$

and

$$N_{\alpha\alpha}(t)=6f^2H^2-2L, \tag{63}$$

$$N_{\alpha\gamma R}(t)=-f^2H^2-\frac{8}{7}f^2H^2\lambda+\frac{2}{7}L\lambda-\frac{4}{7}fH\dot{\lambda}+\frac{1}{2}H^2\sigma_\Phi, \tag{64}$$

$$N_{\alpha\gamma L}(t)=-f^2H^2-\frac{8}{7}f^2H^2\lambda+\frac{2}{7}L\lambda-\frac{4}{7}fH\dot{\lambda}+N_\gamma, \tag{65}$$

$$N_{\gamma\gamma}(t)=-2f^2H^2+\frac{8}{7}f^2H^2\lambda+\frac{4}{7}fH\dot{\lambda}+H^2\left(\sigma_\Phi+\mu_\Phi\right), \tag{66}$$

$$N_\xi(t)=f^2H^2+H^2\left(-\frac{1}{2}\sigma_\Phi+\nu_\Phi\right). \tag{67}$$

where used Eqs. (A24) and (A28), and

$$\dot{f}(t)\;=\;\frac{1}{H}(-2fH^2-L-f^2H^2-f\dot{H}), \tag{68}$$

which follow from the definition of the growth rate $f(t)=d\ln D_+/d\ln a$ and Eq. (31). We can prove that $N_{\alpha\gamma L}(t)$ is equivalent to $N_{\alpha\gamma R}(t)$, using (65) and (64), and $N_\gamma(t)=\frac{1}{2}H^2\sigma_\Phi$, which is demonstrated from Eqs. (A22) and (A29). Then, we write

$$N_{\alpha\gamma}(t)\equiv N_{\alpha\gamma R}(t)=N_{\alpha\gamma L}(t). \tag{69}$$

The general solution of Eq. (61) with (62) is

$$\delta_3(t,\mathbf{p})\;=\;c_+(\mathbf{p})D_+(t)+c_-(\mathbf{p})D_-(t)+\int_0^t\frac{D_-(t)D_+(t')-D_+(t)D_-(t')}{W(t')}S_{\delta 3}(t',\mathbf{p})dt', \tag{70}$$

where $D_+(t)$ and $D_-(t)$ are the growing mode solution and the decaying mode solution, satisfying equation (31), $c_+(\mathbf{p})$ and $c_-(\mathbf{p})$ are integral constants, and $W(t)$ is the Wronskian defined by $W(t)=D_+(t)\dot{D}_-(t)-\dot{D}_+(t)D_-(t)$. Since we assume that the initial density perturbations obey the Gauss distribution, we set $c_\pm(\mathbf{p})=0$, as is done in deriving the second order solution. Then, the solution of the third order density perturbations is given by

$$\begin{aligned} \delta_3(t,\mathbf{p})=D_+^3(t)\Big(&\kappa_{\delta 3}(t)\mathcal{W}_{\alpha\alpha}(\mathbf{p})-\frac{2}{7}\lambda_{\delta 3}(t)\mathcal{W}_{\alpha\gamma R}(\mathbf{p})-\frac{2}{7}\lambda_{\delta 3}(t)\mathcal{W}_{\alpha\gamma L}(\mathbf{p}) \\ &-\frac{2}{21}\mu(t)\mathcal{W}_{\gamma\gamma}(\mathbf{p})+\frac{1}{9}\nu(t)\mathcal{W}_\xi(\mathbf{p})\Big), \end{aligned} \tag{71}$$

where we define

$$\kappa_{\delta 3}(t) = \frac{1}{D_+^3(t)}\int_0^t \frac{D_-(t)D_+(t')-D_+(t)D_-(t')}{W(t')}D_+^3(t')N_{\alpha\alpha}(t')dt', \tag{72}$$

$$\lambda_{\delta 3}(t) = -\frac{7}{2D_+^3(t)}\int_0^t \frac{D_-(t)D_+(t')-D_+(t)D_-(t')}{W(t')}D_+^3(t')N_{\alpha\gamma}(t')dt', \tag{73}$$

$$\mu(t) = -\frac{21}{2D_+^3(t)}\int_0^t \frac{D_-(t)D_+(t')-D_+(t)D_-(t')}{W(t')}D_+^3(t')N_{\gamma\gamma}(t')dt', \tag{74}$$

$$\nu(t) = \frac{9}{D_+^3(t)}\int_0^t \frac{D_-(t)D_+(t')-D_+(t)D_-(t')}{W(t')}D_+^3(t')N_{\xi}(t')dt'. \tag{75}$$

Here note that the parameters in front of $\mathcal{W}_{\alpha\gamma R}(\mathbf{p})$ and $\mathcal{W}_{\alpha\gamma L}(\mathbf{p})$ in expression (71) are the same, which originates from the relation (69). In the limit of the Einstein de Sitter universe in the general relativity, the coefficients, $\kappa_{\delta 3}(t)$, $\lambda_{\delta 3}(t)$, $\mu(t)$, and $\nu(t)$ reduce to 1.

We can redefine these coefficients using the differential equations. Inserting the general form of the solution (71) into (61), we obtain the following differential equations for the coefficients

$$\ddot{\kappa}_{\delta 3}(t) + (6f+2)\dot{\kappa}_{\delta 3}(t) + (6f^2H^2-2L)\kappa_{\delta 3}(t) = N_{\alpha\alpha}(t), \tag{76}$$

$$\ddot{\lambda}_{\delta 3}(t) + (6f+2)\dot{\lambda}_{\delta 3}(t) + (6f^2H^2-2L)\lambda_{\delta 3}(t) = -\frac{7}{2}N_{\alpha\gamma}(t), \tag{77}$$

$$\ddot{\mu}(t) + (6f+2)\dot{\mu}(t) + (6f^2H^2-2L)\mu(t) = -\frac{21}{2}N_{\gamma\alpha}(t), \tag{78}$$

$$\ddot{\nu}(t) + (6f+2)\dot{\nu}(t) + (6f^2H^2-2L)\nu(t) = 9N_{\xi}(t). \tag{79}$$

The homogeneous solution of all these equations is $1/D_+^2(t)$ and $D_-(t)/D_+^3(t)$. Therefore, the differential equations (76) to (79) consistently yield the inhomogeneous solutions (72) to (75), respectively.

We next show that $\kappa_{\delta 3}(t) = 1$ identically. Using the expression (63), we easily find that $\kappa_{\delta 3} = 1$ is the solution of (76). This means that the inhomogeneous solution (72) reduces to $\kappa_{\delta 3} = 1$. We can prove $\kappa_{\delta 3} = 1$ directly from (72), using partial integral.

Furthermore we can show that $\lambda_{\delta 3}(t) = \lambda(t)$ identically. We can rewrite Eq. (77), as follows,

$$\begin{aligned}\ddot{\lambda}_{\delta 3}(t) + (4f+2)H\dot{\lambda}_{\delta 3}(t) + (2f^2H^2-L)\lambda_{\delta 3}(t) + 2fH(\dot{\lambda}_{\delta 3}-\dot{\lambda})&\\ + (4f^2H^2-L)(\lambda_{\delta 3}-\lambda) = \frac{7}{2}(f^2H^2-N_\gamma),&\end{aligned} \tag{80}$$

where we used (69) and (65). We can easily check that $\lambda_{\delta 3}(t)$ and $\lambda(t)$ satisfies the same differential equation (see Eq. (A28)), which leads to $\lambda_{\delta 3}(t) = \lambda(t)$.

In summary, we have the expression equivalent to (71),

$$\delta_3(t,\mathbf{p}) = D_+^3(t)\left(\mathcal{W}_{\alpha\alpha}(\mathbf{p}) - \frac{2}{7}\lambda(t)\mathcal{W}_{\alpha\gamma R}(\mathbf{p}) - \frac{2}{7}\lambda(t)\mathcal{W}_{\alpha\gamma L}(\mathbf{p}) - \frac{2}{21}\mu(t)\mathcal{W}_{\gamma\gamma}(\mathbf{p}) + \frac{1}{9}\nu(t)\mathcal{W}_{\xi}(\mathbf{p})\right). \tag{81}$$

Thus the third order solution of density contrast is characterized by $\lambda(t)$, $\mu(t)$, and $\nu(t)$. Note that $\lambda(t)$ is defined to describe the second order solution, then $\mu(t)$ and $\nu(t)$ are the new coefficients which appear at the third order. Table I summarizes the parameters and the mode-coupling functions necessary to describe the second order solution and the third order solution.

Recently, the authors of [44] investigated the third order solution of the density perturbations, in a similar way, but within a model of the general relativity. In their paper, 6 parameters are introduced to describe the third order density perturbations. Our results suggest that less number of parameters are only independent.

	parameters	mode-coupling functions
$\delta_2(t,\mathbf{p})$	$\lambda(t)$	$\mathcal{W}_\alpha(\mathbf{p})$
$\theta_2(t,\mathbf{p})$	$\lambda_\theta(t)$	$\mathcal{W}_\gamma(\mathbf{p})$
$\delta_3(t,\mathbf{p})$	$\lambda(t),\ \mu(t),\ \nu(t)$	$\mathcal{W}_{\alpha\alpha}(\mathbf{p}),\ \mathcal{W}_{\alpha\gamma R}(\mathbf{p}),\ \mathcal{W}_{\alpha\gamma L}(\mathbf{p})$
$\theta_3(t,\mathbf{p})$	$\lambda(t),\ \lambda_\theta(t),\ \mu_\theta(t),\ \nu_\theta(t)$	$\mathcal{W}_{\gamma\gamma}(\mathbf{p}),\ \mathcal{W}_\xi(\mathbf{p})$

TABLE I: Functions for the mode-couplings and parameters necessary to describe the second order solution and the third order solution.

Inserting the solution (81) into Eq. (59), we find the solution for the velocity divergence

$$\theta_3(t,\mathbf{p}) = -D_+^3(t)f\bigg(\mathcal{W}_{\alpha\alpha}(\mathbf{p}) - \frac{4}{7}\lambda_\theta(t)\mathcal{W}_{\alpha\gamma R} - \frac{2}{7}\lambda(t)\mathcal{W}_{\alpha\gamma L}(\mathbf{p}) - \frac{2}{7}\mu_\theta(t)\mathcal{W}_{\gamma\gamma}(\mathbf{p}) + \frac{1}{3}\nu_\theta(t)\mathcal{W}_\xi(\mathbf{p})\bigg), \tag{82}$$

where we define

$$\mu_\theta(t) = \mu(t) + \frac{\dot{\mu}(t)}{3fH}, \tag{83}$$

$$\nu_\theta(t) = \nu(t) + \frac{\dot{\nu}(t)}{3fH}. \tag{84}$$

Here note that $\lambda_\theta(t)$ is the parameter to describe the second order solution, and $\mu_\theta(t)$ and $\nu_\theta(t)$ are the new parameters which appear at the third order.

In summary, we first introduced *nine* mode-coupling functions in the third order equations, (54) and (55) with (49). We find the *three* identities (56), (57) and (58). Then, only *six* mode-coupling functions are independent in the *nine* ones. This conclusion that the number of the linearly independent mode-coupling functions is *six* can be proved by using the generalized Wronskian. The coefficients in front of $\mathcal{W}_{\alpha\alpha R}$ and $\mathcal{W}_{\alpha\alpha L}$ in equation (54) are the same, which leads to the final third order solution (81) and (82) expressed in terms of the *five* mode-coupling functions.

IV. POWER SPECTRUM

The third order solution of the density perturbations enable one to compute the 1-loop (second order) power spectrum. The second order matter power spectrum was computed by many authors [27–33, 35–37, 44], in general relativity and modified gravity models. We find the expression for the 1-loop order the power spectra of density contrast and velocity divergence by

$$\langle\delta(t,\mathbf{k}_1)\delta(t,\mathbf{k}_2)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_{\delta\delta}(t,k), \tag{85}$$

$$\langle\delta(t,\mathbf{k}_1)\theta(t,\mathbf{k}_2)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)(-f)P_{\delta\theta}(t,k), \tag{86}$$

$$\langle\theta(t,\mathbf{k}_1)\theta(t,\mathbf{k}_2)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)f^2P_{\theta\theta}(t,k), \tag{87}$$

where we use the notation $k = |\mathbf{k}_1|$. Some details of their derivations are described in appendix C, we here show the results,

$$P_{\delta\delta}(t,k) = D_+^2(t)P_{\rm L}(k) + D_+^4(t)\left(P_{\delta\delta}^{(22)}(t,k) + 2P_{\delta\delta}^{(13)}(t,k)\right), \tag{88}$$

$$P_{\delta\theta}(t,k) = D_+^2(t)P_{\rm L}(k) + D_+^4(t)\left(P_{\delta\theta}^{(22)}(t,k) + 2P_{\delta\theta}^{(13)}(t,k)\right), \tag{89}$$

$$P_{\theta\theta}(t,k) = D_+^2(t)P_{\rm L}(k) + D_+^4(t)\left(P_{\theta\theta}^{(22)}(t,k) + 2P_{\theta\theta}^{(13)}(t,k)\right), \tag{90}$$

where $D_+^2(t)P_{\rm L}(k)$ is the linear matter power spectrum, and we define

$$P^{(22)}_{\delta\delta}(t,k) = \frac{k^3}{98(2\pi)^2}\int drP_{\rm L}(rk)\int_{-1}^{1} dxP_{\rm L}(k(1+r^2-2rx)^{1/2}) \times \frac{((7-4\lambda)r+7x+2(2\lambda-7)rx^2)^2}{(1+r^2-2rx)^2}, \tag{91}$$

$$P^{(22)}_{\delta\theta}(t,k) = \frac{k^3}{98(2\pi)^2}\int drP_{\rm L}(rk)\int_{-1}^{1} dxP_{\rm L}(k(1+r^2-2rx)^{1/2}) \times \frac{((7-4\lambda)r+7x+2(2\lambda-7)rx^2)((7-8\lambda_\theta)r+7x+2(4\lambda_\theta-7)rx^2)}{(1+r^2-2rx)^2} \tag{92}$$

$$P^{(22)}_{\theta\theta}(t,k) = \frac{k^3}{98(2\pi)^2}\int drP_{\rm L}(rk)\int_{-1}^{1} dxP_{\rm L}(k(1+r^2-2rx)^{1/2}) \times \frac{((7-8\lambda_\theta)r+7x+2(4\lambda_\theta-7)rx^2)^2}{(1+r^2-2rx)^2}, \tag{93}$$

and

$$2P^{(13)}_{\delta\delta}(t,k) = \frac{k^3}{252(2\pi)^2}P_{\rm L}(k)\int drP_{\rm L}(rk)\biggl[12\mu\frac{1}{r^2}-2(21+36\lambda+22\mu)+4(84-48\lambda-11\mu)r^2 - 6(21-12\lambda-2\mu)r^4+\frac{3}{r^3}\left(r^2-1\right)^3\left((21-12\lambda-2\mu)r^2+2\mu\right)\ln\left(\frac{r+1}{|r-1|}\right)\biggr], \tag{94}$$

$$2P^{(13)}_{\delta\theta}(t,k) = \frac{k^3}{252(2\pi)^2}P_{\rm L}(k)\int drP_{\rm L}(rk)\biggl[6(\mu+3\mu_\theta)\frac{1}{r^2}-2(21+36\lambda+11\mu+33\mu_\theta) +2(168-96\lambda-11\mu-33\mu_\theta)r^2-6(21-12\lambda-\mu-3\mu_\theta)r^4 +\frac{3}{r^3}\left(r^2-1\right)^3\left((21-12\lambda-\mu-3\mu_\theta)r^2+(\mu+3\mu_\theta)\right)\ln\left(\frac{r+1}{|r-1|}\right)\biggr], \tag{95}$$

$$2P^{(13)}_{\theta\theta}(t,k) = \frac{k^3}{84(2\pi)^2}P_{\rm L}(k)\int drP_{\rm L}(rk)\biggl[12\mu_\theta\frac{1}{r^2}-2(7+12\lambda+22\mu_\theta)+4(28-16\lambda-11\mu_\theta)r^2 -6(7-4\lambda-2\mu_\theta)r^4+\frac{3}{r^3}\left(r^2-1\right)^3\left((7-4\lambda-2\mu_\theta)r^2+2\mu_\theta\right)\ln\left(\frac{r+1}{|r-1|}\right)\biggr]. \tag{96}$$

The third order solutions of the density contrast and the velocity divergence are described by 6 parameters in Table I. The 1-loop power spectra are described by 4 parameters, and they do not depend on $\nu(t)$ and $\nu_\theta(t)$ (see Table II). In deriving the 1-loop power spectrum, we find that the relation,

$$\xi(\mathbf{k},\mathbf{q}_1,-\mathbf{q}_1)=0, \tag{97}$$

holds, which prevents the 1-loop power spectrum from depending on $\nu(t)$ and $\nu_\theta(t)$. Details are described in appendix C and D.

	parameters
$P_{\delta\delta}$	$\lambda(t),\ \mu(t)$
$P_{\delta\theta}$	$\lambda(t),\ \mu(t),\ \lambda_\theta(t),\ \mu_\theta(t)$
$P_{\theta\theta}$	$\lambda(t),\ \lambda_\theta(t),\ \mu_\theta(t)$

TABLE II: Summary of the parameters to characterize the 1-loop order power spectra $P_{\delta\delta}$, $P_{\delta\theta}$, and $P_{\theta\theta}$, respectively.

V. TRISPECTRUM

Here we present the expression for the matter trispectrum in the real space, which is defined by

$$\langle\delta(t,\mathbf{k}_1)\delta(t,\mathbf{k}_2)\delta(t,\mathbf{k}_3)\delta(t,\mathbf{k}_4)\rangle_c=(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)T(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4). \tag{98}$$

Using the solution up to the third order of the density perturbations, we find

$$T(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=D_+^6(t)\Big(T_{1122}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)+T_{1113}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\Big), \tag{99}$$

where we define

$$\begin{aligned}T_{1122}(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)&=4P_{11}(k_1)P_{11}(k_2)\left[P_{11}(|\mathbf{k}_1+\mathbf{k}_3|)F_2(t,\mathbf{k}_1,-\mathbf{k}_1-\mathbf{k}_3)F_2(t,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_3)\right.\\&\quad\left.+P_{11}(|\mathbf{k}_1+\mathbf{k}_4|)F_2(t,\mathbf{k}_1,-\mathbf{k}_1-\mathbf{k}_4)F_2(t,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_4)\right]+5\text{ cyclic terms,}\end{aligned} \tag{100}$$

$$T_{1113}(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)=6P_{11}(k_1)P_{11}(k_2)P_{11}(k_3)F_3(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+4\text{ cyclic terms.} \tag{101}$$

Note that T_{1122} contains F_2, defined by Eq. (C7), which depends on $\lambda(t)$, while T_{1113} contains F_3, defined by Eq. (C9), which depends on $\lambda(t)$, $\mu(t)$ and $\nu(t)$. Therefore, the matter trispectrum depends on these three parameters.

VI. APPLICATION OF KGB MODEL

In this section, we exemplify the effect of the modified gravity on the 1-loop power spectrum. We here consider the kinetic gravity braiding (KGB) model [60, 66], which is considered in Ref. [26] to demonstrate the effect of the modified gravity on the bispectrum. We briefly review the model. The action of the KGB model is written as

$$S=\int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R+K(\phi,X)-G_3(\phi,X)\Box\phi+\mathcal{L}_{\rm m}\right], \tag{102}$$

where $M_{\rm pl}$ is the Planck mass, which is related with the gravitational constant G_N by $8\pi G_N=1/M_{\rm pl}^2$. Comparing this action (102) with that of the most general second-order scalar-tensor theory (1), the action of the kinetic gravity braiding model is produced by setting

$$G_4=\frac{M_{\rm pl}^2}{2},\qquad G_5=0, \tag{103}$$

and we choose K and G_3 as

$$K=-X,\qquad G_3=M_{\rm pl}\left(\frac{r_c^2}{M_{\rm pl}^2}X\right)^n, \tag{104}$$

where n and r_c are the model parameters. Useful expressions of the kinetic gravity braiding model are summarized in appendix E.

When we consider the attractor solution, which satisfies $3\dot\phi HG_{3X}=1$, the Friedmann equation is written in the form

$$\left(\frac{H}{H_0}\right)^2=\frac{\Omega_0}{a^3}+(1-\Omega_0)\left(\frac{H}{H_0}\right)^{-2/(2n-1)}, \tag{105}$$

where H_0 is the Hubble constant and Ω_0 is the density parameter at the present time, and the model parameters must satisfy

$$H_0r_c=\left(\frac{2^{n-1}}{3n}\right)^{1/2n}\left[\frac{1}{6(1-\Omega_0)}\right]^{(2n-1)/4n}. \tag{106}$$

On the attractor solution, $L(t)$, defined by Eq. (32), reduces to

$$L(t) = -\frac{3}{2}\frac{2n + (3n-1)\Omega_{\rm m}}{5n - \Omega_{\rm m}}H^2, \tag{107}$$

where $\Omega_{\rm m}$ is defined by $\Omega_{\rm m}(a) = \Omega_0 H_0^2/H(a)^2a^3$. The linear growth factor D_+ is obtained from Eq. (31) with (107) and (105). However, note that the quasi-static approximation on the scales of the large scale structures holds for $n \lesssim 10$ (see [66]).

The second order solution and the third order solution are obtained with

$$N_\gamma(t) = \frac{1}{2}H^2\sigma_\Phi(t) = -\frac{9}{4}\frac{(1-\Omega_{\rm m})(2n-\Omega_{\rm m})^3}{\Omega_{\rm m}(5n-\Omega_{\rm m})^3}H^2, \tag{108}$$

$$H^2\mu_\Phi(t) = \frac{9(1-\Omega_{\rm m})(2n-\Omega_{\rm m})^3(4n^2(21+25\lambda\Omega_{\rm m}) - 4n\Omega_{\rm m}(21+10\lambda\Omega_{\rm m}) + \Omega_{\rm m}^2(21+4\lambda\Omega_{\rm m}))}{28\Omega_{\rm m}^2(5n-\Omega_{\rm m})^5}H^2, \tag{109}$$

$$\nu_\Phi(t) = 0. \tag{110}$$

We have $\lambda(t)$ from (A23) with (108). Using these results and Eqs. (66) and (67), we have the expressions for $\mu(t)$ and $\nu(t)$ from (74) and (75). Eqs. (A24), (83), and (84) give expressions for $\lambda_\theta(t),\ \mu_\theta(t),\ \nu_\theta(t)$, respectively.

Table III lists the numerical values of these variables at the redshift $z = 1$, 0.5 and 0, for the KGB model with $n = 1$, 2, 5, as well as the ΛCDM model.

Figure 1 shows $\lambda,\ \mu,\ \nu,\ \lambda_\theta,\ \mu_\theta,\ \nu_\theta$ as function of the scale factor a. In each panel, the blue dash-dotted curve is the ΛCDM model, and the red dotted curve, the yellow dashed curve, and the green thick solid curve are the KGB model with $n = 1,\ 2$, and 5, respectively. All the curves take the limiting value unity at $a = 0$, but deviate from the unity as a evolves. Note that the deviation of $\lambda,\ \mu,\ \nu$ from unity is small, of the order of a few percent, but the deviation of $\lambda_\theta,\ \nu_\theta$ is rather large, which could be 10 percent. This is because the parameters associated with the velocity, λ_θ and ν_θ defined by Eqs. (A24) and (84), respectively, contain the time derivative term, which makes a large contribution. Plus, some part of the difference between the ΛCDM and the KGB model come from the difference of the growth rate f. Deviation of μ_θ in the KGB model from that in the ΛCDM model is rather small compared with the deviations of λ_θ and ν_θ, which comes from the fact that μ is not a monotonic increasing function but there exists a maximum value at $a \lesssim 1$.

Figure 2 shows the 1-loop power spectra $P_{\delta\delta},\ P_{\delta\theta},\ P_{\theta\theta}$, from the top to the bottom, respectively, which are normalized by those of the ΛCDM model. These are the snapshots at $z = 0$, and we adopted the same normalization begin $\sigma_8 = 0.8$ for each model, which means that all the models have the same linear matter power spectrum. In each panel, the red dotted curve, the yellow dashed curve, and the green thick curve

	ΛCDM	KGB($n=1$)	KGB($n=2$)	KGB($n=5$)
$D_+(z=1\ /\ 0.5\ /\ 0)$	0.477 / 0.602 / 0.779	0.496 / 0.642 / 0.858	0.489 / 0.628 / 0.838	0.484 / 0.620 / 0.827
$f(z=1\ /\ 0.5\ /\ 0)$	0.869 / 0.749 / 0.513	0.951 / 0.835 / 0.593	0.919 / 0.813 / 0.605	0.904 / 0.805 / 0.612
$\lambda(z=1\ /\ 0.5\ /\ 0)$	0.999 / 0.997 / 0.994	1.000 / 0.999 / 1.003	1.000 / 1.000 / 1.011	1.000 / 1.002 / 1.019
$\mu(z=1\ /\ 0.5\ /\ 0)$	0.999 / 0.998 / 0.996	1.000 / 1.001 / 1.015	1.001 / 1.005 / 1.015	1.003 / 1.007 / 1.011
$\nu(z=1\ /\ 0.5\ /\ 0)$	0.998 / 0.996 / 0.991	1.000 / 0.999 / 1.014	1.000 / 1.003 / 1.034	1.002 / 1.008 / 1.049
$\lambda_\theta(z=1\ /\ 0.5\ /\ 0)$	0.994 / 0.991 / 0.983	0.998 / 0.995 / 1.043	0.999 / 1.004 / 1.073	1.003 / 1.014 / 1.095
$\mu_\theta(z=1\ /\ 0.5\ /\ 0)$	0.997 / 0.995 / 0.991	1.000 / 1.006 / 1.041	1.006 / 1.018 / 1.008	1.010 / 1.021 / 0.974
$\nu_\theta(z=1\ /\ 0.5\ /\ 0)$	0.994 / 0.990 / 0.980	0.998 / 0.998 / 1.089	1.002 / 1.014 / 1.136	1.009 / 1.030 / 1.169

TABLE III: Numerical values of the growth factor D_+, the linear growth rate f, and the coefficients $\lambda,\ \mu,\ \nu,\ \lambda_\theta,\ \mu_\theta,\ \nu_\theta$ at the redshifts $z = 1.0,\ 0.5$ and 0, for the ΛCDM mode and the KGB model with $n = 1,\ 2,\ 5$. In each cell, a set of the three numerics means the values at the redshift $z = 1.0,\ 0.5$ and 0 from left to right, respectively.

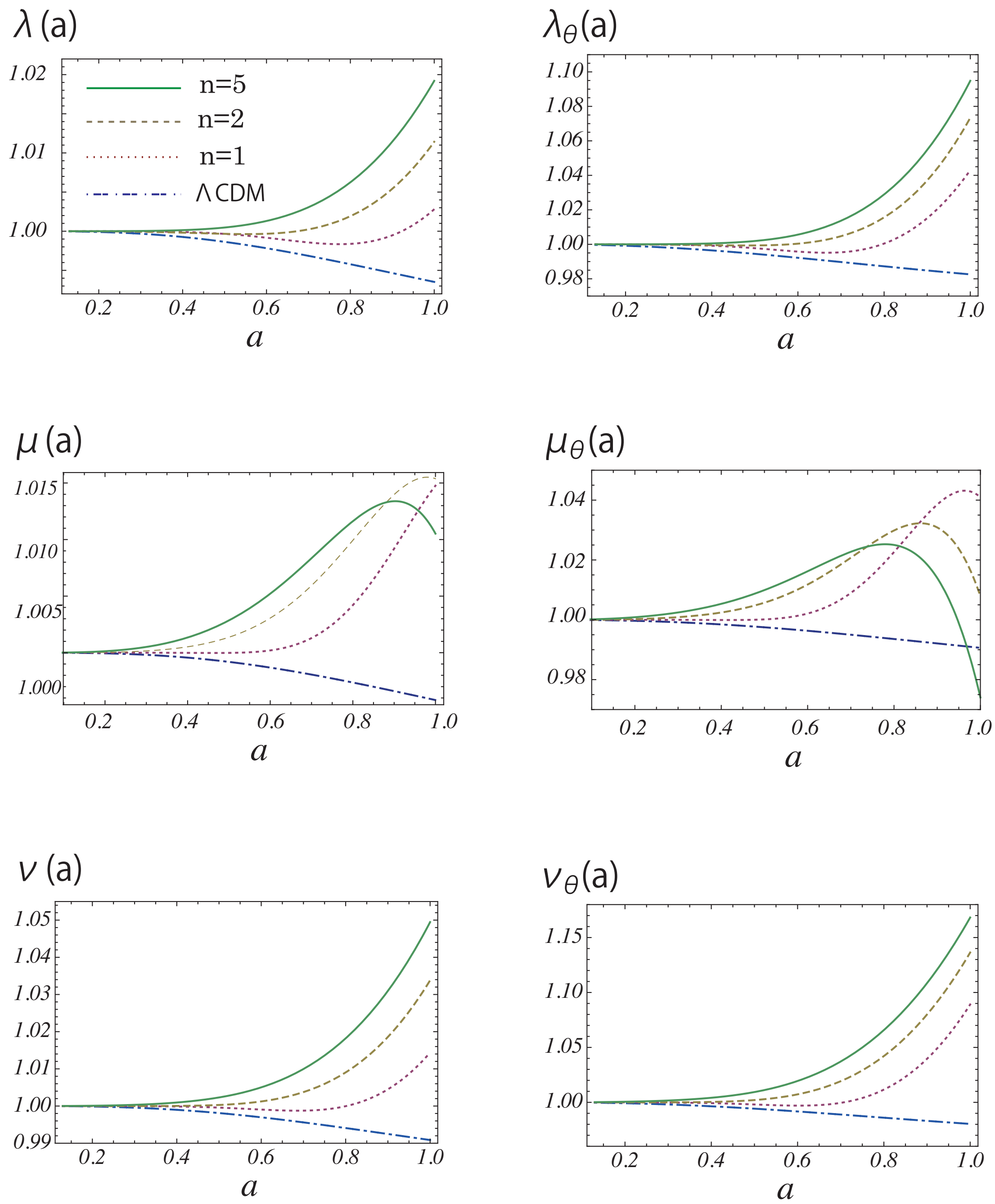

FIG. 1: λ, μ, ν, λ_θ, μ_θ, ν_θ as function of the scale factor a. In each panel, the blue dash-dotted curve is the ΛCDM model, and the red dotted curve, the yellow dashed curve, and the green thick solid curve are the KGB model with $n = 1$, 2, and 5, respectively.

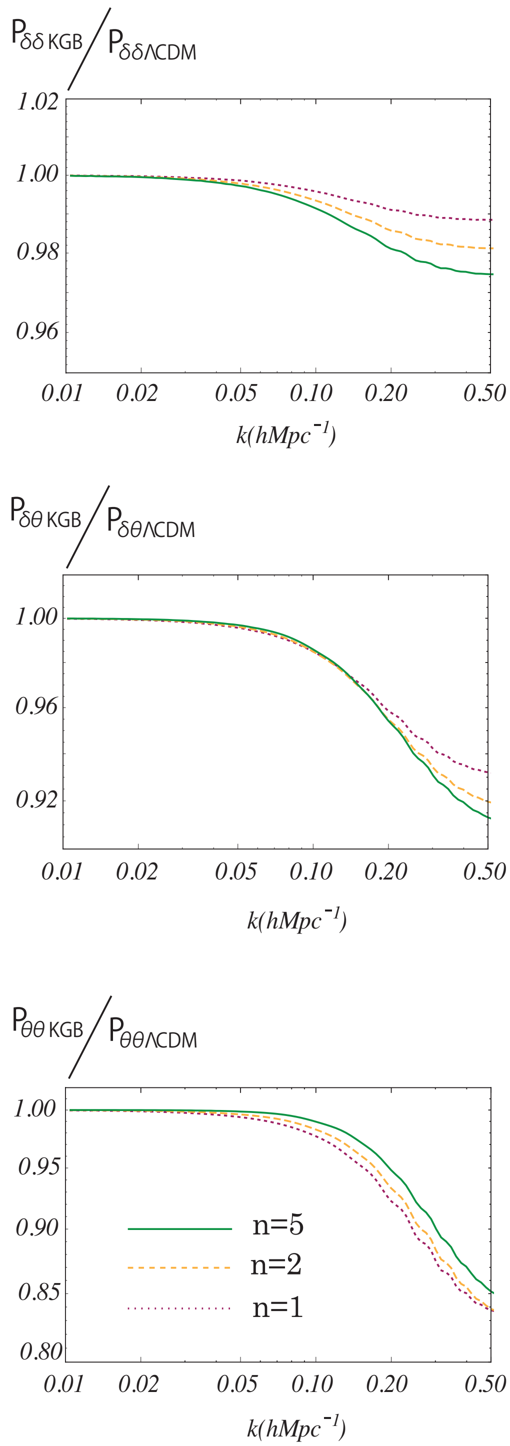

FIG. 2: Relative deviation of the power spectra $P_{\delta\delta}(k)$ (top panel), $P_{\delta\theta}(k)$ (middle pane.), $P_{\theta\theta}(k)$ (bottom panel), under the kinetic gravity braiding model with $n = 1$ (red dotted curve), $n = 2$ (yellow dashed curve), $n = 5$ (green thick curve), which are divided by those under the ΛCDM model. The panels show the snapshot at the redshift $z = 0$.

show the KGB model with $n = 1$, 2 and 5, respectively. In the linear regime $k \lesssim 0.1[h\mathrm{Mpc}^{-1}]$, all the models converge because they have the same linear matter power spectrum due to the same normalization $\sigma_8 = 0.8$. The differences between the KGB model and the ΛCDM model appear for the quasi-nonlinear regime $k \gtrsim 0.1[h\mathrm{Mpc}^{-1}]$ due to the nonlinear effect. Because all the model have the same linear matter power spectrum, this figure shows that the enhancement of the power spectrum due to the nonlinear effect is small in the KGB model compared with that in the ΛCDM model. This is understood as the results of the Vainshtein effect. Furthermore, the deviation from the ΛCDM model is more significant in the velocity power spectrum than that in the density power spectrum. In general, the amplitude of the 1-loop power spectra $P_{\delta\delta}$, $P_{\delta\theta}$, and $P_{\theta\theta}$ are decreased when any of $\lambda(t)$, $\mu(t)$, $\lambda_\theta(t)$, and $\mu_\theta(t)$ is increased. The behavior of $P_{\delta\theta}$ and $P_{\theta\theta}$ in the quasi-nonlinear regime is dominantly influenced by $\lambda_\theta(t)$ and $\mu_\theta(t)$.

VII. SUMMARY AND CONCLUSIONS

We found the third order solutions of the cosmological density perturbations in the Horndeski's most general scalar-tensor theory assuming that the Vainshtein mechanism is at work. We solved the equations under the quasi-static approximation, and the solutions describe the quasi-nonlinear aspects of the cosmological density contrast and the velocity divergence under the Vainshtein mechanism. In this work, we thoroughly investigate the independence property of the mode-couplings functions describing the non-linear interactions. We found that the third order solution of the density contrast is characterized by 3 parameters for the nonlinear interactions, one of which is the same as that for the second order solutions. The third order solution of the velocity divergence is characterized by 4 parameters for the nonlinear interactions, two of which are the same parameters as those of the second order solutions. The nonlinear features of the perturbative solutions up to the third order are characterized by 6 parameters. Furthermore, the 1-loop order power spectra obtained with the third order solutions are described by 4 parameters. Assuming the KGB model, we demonstrated the effect of the modified gravity in the 1-loop order power spectra at the quantitative level. We found that the deviation from the ΛCDM model appears in the power spectra of the density contrast and the velocity divergence, which can be understood as the results of the Vainshtein mechanism. The deviation from the ΛCDM model is more significant in the velocity divergence than the density contrast, which is explained by a dominant contribution of the parameters $\lambda_\theta(t)$ and $\mu_\theta(t)$. It is interesting to investigate whether this is a general feature of the modified gravity with the Vainshtein mechanism or not.

Acknowledgment

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Appendix A: Coefficients in equations and solutions

We summarize the definitions of the coefficients in the gravitational and scalar field equations (5), (6), (9).

$$A_0 = \frac{\dot{\Theta}}{H^2} + \frac{\Theta}{H} + \mathcal{F}_T - 2\mathcal{G}_T - 2\frac{\dot{\mathcal{G}}_T}{H} - \frac{\mathcal{E}+\mathcal{P}}{2H^2}, \tag{A1}$$

$$A_1 = \frac{\dot{\mathcal{G}}_T}{H} + \mathcal{G}_T - \mathcal{F}_T, \tag{A2}$$

$$A_2 = \mathcal{G}_T - \frac{\Theta}{H}, \tag{A3}$$

$$\begin{aligned} B_0 = \frac{X}{H}\Big\{ & \dot{\phi}G_{3X} + 3\left(\dot{X}+2HX\right)G_{4XX} + 2X\dot{X}G_{4XXX} - 3\dot{\phi}G_{4\phi X} + 2\dot{\phi}XG_{4\phi XX} \\ & + \left(\dot{H}+H^2\right)\dot{\phi}G_{5X} + \dot{\phi}\left[2H\dot{X} + \left(\dot{H}+H^2\right)X\right]G_{5XX} + H\dot{\phi}X\dot{X}G_{5XXX} \\ & -2\left(\dot{X}+2HX\right)G_{5\phi X} - \dot{\phi}XG_{5\phi\phi X} - X\left(\dot{X}-2HX\right)G_{5\phi XX}\Big\}, \end{aligned} \tag{A4}$$

$$B_1 = 2X\left[G_{4X} + \ddot{\phi}\left(G_{5X} + XG_{5XX}\right) - G_{5\phi} + XG_{5\phi X}\right], \tag{A5}$$

$$B_2 = -2X\left(G_{4X} + 2XG_{4XX} + H\dot{\phi}G_{5X} + H\dot{\phi}XG_{5XX} - G_{5\phi} - XG_{5\phi X}\right), \tag{A6}$$

$$B_3 = H\dot{\phi}XG_{5X}, \tag{A7}$$

$$C_0 = 2X^2G_{4XX} + \frac{2X^2}{3}\left(2\ddot{\phi}G_{5XX} + \ddot{\phi}XG_{5XXX} - 2G_{5\phi X} + XG_{5\phi XX}\right), \tag{A8}$$

$$C_1 = H\dot{\phi}X\left(G_{5X} + XG_{5XX}\right), \tag{A9}$$

where we define

$$\mathcal{F}_T = 2\left[G_4 - X\left(\ddot{\phi}G_{5X} + G_{5\phi}\right)\right], \tag{A10}$$

$$\mathcal{G}_T = 2\left[G_4 - 2XG_{4X} - X\left(H\dot{\phi}G_{5X} - G_{5\phi}\right)\right], \tag{A11}$$

$$\begin{aligned} \Theta = & -\dot{\phi}XG_{3X} + 2HG_4 - 8HXG_{4X} - 8HX^2G_{4XX} + \dot{\phi}G_{4\phi} + 2X\dot{\phi}G_{4\phi X} \\ & -H^2\dot{\phi}\left(5XG_{5X} + 2X^2G_{5XX}\right) + 2HX\left(3G_{5\phi} + 2XG_{5\phi X}\right), \end{aligned} \tag{A12}$$

$$\begin{aligned} \mathcal{E} = & \, 2XK_X - K + 6X\dot{\phi}HG_{3X} - 2XG_{3\phi} - 6H^2G_4 + 24H^2X(G_{4X} + XG_{4XX}) \\ & -12HX\dot{\phi}G_{4\phi X} - 6H\dot{\phi}G_{4\phi} + 2H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) \\ & -6H^2X(3G_{5\phi} + 2XG_{5\phi X}), \end{aligned} \tag{A13}$$

$$\begin{aligned} \mathcal{P} = & \, K - 2X(G_{3\phi} + \ddot{\phi}G_{3X}) + 2(3H^2 + 2\dot{H})G_4 - 12H^2XG_{4X} - 4H\dot{X}G_{4X} \\ & -8\dot{H}XG_{4X} - 8HX\dot{X}G_{4XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4\phi} + 4XG_{4\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4\phi X} \\ & -2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5X} - 4H^2X^2\ddot{\phi}G_{5XX} + 4HX(\dot{X} - HX)G_{5\phi X} \\ & +2\left[2(HX)^{\cdot} + 3H^2X\right]G_{5\phi} + 4HX\dot{\phi}G_{5\phi\phi}. \end{aligned} \tag{A14}$$

The coefficients in the first and the second order solutions are defined as follows,

$$\mathcal{R}(t) = A_0\mathcal{F}_T - A_1^2, \tag{A15}$$
$$\mathcal{S}(t) = A_0\mathcal{G}_T + A_1A_2, \tag{A16}$$
$$\mathcal{T}(t) = A_1\mathcal{G}_T + A_2\mathcal{F}_T, \tag{A17}$$
$$\mathcal{Z}(t) = 2\left(A_0\mathcal{G}_T^2 + 2A_1A_2\mathcal{G}_T + A_2^2\mathcal{F}_T\right), \tag{A18}$$
$$\kappa_\Phi(t) = \frac{\rho_\mathrm{m}\mathcal{R}}{H^2\mathcal{Z}}, \tag{A19}$$
$$\kappa_\Psi(t) = \frac{\rho_\mathrm{m}\mathcal{S}}{H^2\mathcal{Z}}, \tag{A20}$$
$$\kappa_Q(t) = \frac{\rho_\mathrm{m}\mathcal{T}}{H^2\mathcal{Z}}, \tag{A21}$$
$$N_\gamma(t) = \frac{H^4}{\rho_\mathrm{m}}\left(2B_0\kappa_Q^3 - 3B_1\kappa_\Psi\kappa_Q^2 - 3B_2\kappa_\Phi\kappa_Q^2 - 6B_3\kappa_\Phi\kappa_\Psi\kappa_Q\right), \tag{A22}$$
$$\lambda(t) = \frac{7}{2D_+^2(t)}\int_0^t \frac{D_-(t)D_+(t') - D_+(t)D_-(t')}{W(t')}D_+^2(t')\left(f^2H^2 - N_\gamma(t')\right)dt', \tag{A23}$$
$$\lambda_\theta(t) = \lambda(t) + \frac{\dot{\lambda}(t)}{2fH}, \tag{A24}$$
$$\lambda_\Phi(t) = -\frac{2}{7}\kappa_\Phi\lambda(t) + \frac{1}{\mathcal{Z}}\left(2B_0\mathcal{T}\kappa_Q^2 - 3B_1\mathcal{S}\kappa_Q^2 - 3B_2\mathcal{R}\kappa_Q^2 - 6B_3\mathcal{R}\kappa_\Psi\kappa_Q\right), \tag{A25}$$
$$\begin{aligned}\lambda_\Psi(t) = -\frac{2}{7}\kappa_\Psi\lambda(t) + \frac{1}{\mathcal{Z}}\Big(&2B_0A_2\mathcal{G}_T\kappa_Q^2 + B_1(A_2^2\kappa_Q^2 - 2A_2\mathcal{G}_T\kappa_\Psi\kappa_Q) - B_2(\mathcal{S}\kappa_Q^2 + 2A_2\mathcal{G}_T\kappa_\Phi\kappa_Q)\\ &-2B_3(\mathcal{S}\kappa_\Psi\kappa_Q - A_2^2\kappa_\Phi\kappa_Q + A_2\mathcal{G}_T\kappa_\Phi\kappa_\Psi)\Big),\end{aligned} \tag{A26}$$
$$\begin{aligned}\lambda_Q(t) = -\frac{2}{7}\kappa_Q\lambda(t) - \frac{1}{\mathcal{Z}}\Big(&2B_0\mathcal{G}_T^2\kappa_Q^2 + B_1(A_2\mathcal{G}_T\kappa_Q^2 - 2\mathcal{G}_T^2\kappa_\Psi\kappa_Q) + B_2(\mathcal{T}\kappa_Q^2 - 2\mathcal{G}_T^2\kappa_\Phi\kappa_Q)\\ &+2B_3(\mathcal{T}\kappa_\Psi\kappa_Q + A_2\mathcal{G}_T\kappa_\Phi\kappa_Q - \mathcal{G}_T^2\kappa_\Phi\kappa_\Psi)\Big).\end{aligned} \tag{A27}$$

Some details are described in the previous paper [26], but one can show that $\lambda(t)$ obeys the differential equation,

$$\ddot{\lambda}(t) + (4f+2)H\dot{\lambda}(t) + (2f^2H^2 - L)\lambda(t) = \frac{7}{2}(f^2H^2 - N_\gamma). \tag{A28}$$

The coefficients for the third order solutions are defined as

$$\sigma_\Phi(t) = \frac{2}{\mathcal{Z}}\left(2B_0\mathcal{T}\kappa_Q^2 - 3B_1\mathcal{S}\kappa_Q^2 - 3B_2\mathcal{R}\kappa_Q^2 - 6B_3\mathcal{R}\kappa_\Psi\kappa_Q\right), \tag{A29}$$

$$\mu_\Phi(t) = \frac{2}{\mathcal{Z}}\left(2B_0\mathcal{T}\kappa_Q\lambda_Q - B_1\left(2\mathcal{S}\kappa_Q\lambda_Q + \mathcal{T}\kappa_Q\lambda_\Psi\right) - B_2\left(2\mathcal{R}\kappa_Q\lambda_Q + \mathcal{T}\kappa_Q\lambda_\Phi\right)\right.$$
$$\left. -2B_3\left(\mathcal{R}\kappa_\Psi\lambda_Q + \mathcal{R}\kappa_Q\lambda_\Psi + \mathcal{S}\kappa_Q\lambda_\Phi\right)\right), \tag{A30}$$

$$\nu_\Phi(t) = \frac{2}{3\mathcal{Z}}\left(-3C_0\mathcal{T}\kappa_Q^3 - 4C_1\mathcal{R}\kappa_Q^3\right), \tag{A31}$$

$$\sigma_\Psi(t) = \frac{2}{\mathcal{Z}}\left(2B_0A_2\mathcal{G}_T\kappa_Q^2 - B_1\left(2A_2\mathcal{G}_T\kappa_\Psi\kappa_Q - A_2^2\kappa_Q^2\right) - B_2\left(\mathcal{S}\kappa_Q^2 + 2A_2\mathcal{G}_T\kappa_\Phi\kappa_Q\right)\right.$$
$$\left. -2B_3\left(\mathcal{S}\kappa_\Psi\kappa_Q + A_2\mathcal{G}_T\kappa_\Phi\kappa_\Psi - A_2^2\kappa_\Phi\kappa_Q\right)\right), \tag{A32}$$

$$\mu_\Psi(t) = \frac{2}{\mathcal{Z}}\left(2B_0A_2\mathcal{G}_T\kappa_Q\lambda_Q - B_1\left(A_2\mathcal{G}_T\left(\kappa_\Psi\lambda_Q + \kappa_Q\lambda_\Psi\right) - A_2^2\kappa_Q\lambda_Q\right)\right.$$
$$-B_2\left(\mathcal{S}\kappa_Q\lambda_Q + A_2\mathcal{G}_T\left(\kappa_\Phi\lambda_Q + \kappa_Q\lambda_\Phi\right)\right)$$
$$\left. -B_3\left(\mathcal{S}\left(\kappa_\Psi\lambda_Q + \kappa_Q\lambda_\Psi\right) + A_2\mathcal{G}_T\left(\kappa_\Phi\lambda_\Psi + \kappa_\Psi\lambda_\Phi\right) - A_2^2\left(\kappa_\Phi\lambda_Q + \kappa_Q\lambda_\Phi\right)\right)\right), \tag{A33}$$

$$\nu_\Psi(t) = \frac{2}{3\mathcal{Z}}\left(-3C_0A_2\mathcal{G}_T\kappa_Q^3 - C_1\left(\mathcal{S}\kappa_Q^3 + 3A_2\mathcal{G}_T\kappa_\Phi\kappa_Q^2\right)\right), \tag{A34}$$

$$\sigma_Q(t) = \frac{2}{\mathcal{Z}}\left(-2B_0\mathcal{G}_T^2\kappa_Q^2 + B_1\left(2\mathcal{G}_T^2\kappa_\Psi\kappa_Q - A_2\mathcal{G}_T\kappa_Q^2\right) - B_2\left(\mathcal{T}\kappa_Q^2 - 2\mathcal{G}_T^2\kappa_\Phi\kappa_Q\right)\right.$$
$$\left. -2B_3\left(\mathcal{S}\kappa_Q^2 - \mathcal{G}_T^2\kappa_\Phi\kappa_\Psi + A_2\mathcal{G}_T\kappa_\Phi\kappa_Q\right)\right), \tag{A35}$$

$$\mu_Q(t) = \frac{2}{\mathcal{Z}}\left(-2B_0\mathcal{G}_T^2\kappa_Q\lambda_Q + B_1\left(\mathcal{G}_T^2\left(\kappa_\Psi\lambda_Q + \kappa_Q\lambda_\Psi\right) - A_2\mathcal{G}_T\kappa_Q\lambda_Q\right)\right.$$
$$-B_2\left(\mathcal{T}\kappa_Q\lambda_Q - \mathcal{G}_T^2\left(\kappa_\Phi\lambda_Q + \kappa_Q\lambda_\Phi\right)\right)$$
$$\left. -B_3\left(\left(\mathcal{S}\kappa_Q\lambda_Q + \mathcal{T}\kappa_Q\lambda_\Psi\right) - \mathcal{G}_T^2\left(\kappa_\Phi\lambda_\Psi + \kappa_\Psi\lambda_\Phi\right) + A_2\mathcal{G}_T\left(\kappa_\Phi\lambda_Q + \kappa_Q\lambda_\Phi\right)\right)\right), \tag{A36}$$

$$\nu_Q(t) = \frac{2}{3\mathcal{Z}}\left(3C_0\mathcal{G}_T^2\kappa_Q^3 + C_1\left(-\mathcal{T}\kappa_Q^3 + 3\mathcal{G}_T^2\kappa_\Phi\kappa_Q^2\right)\right). \tag{A37}$$

Appendix B: The third order mode-coupling functions

In this appendix, we summarize the functions that describe the nonlinear mode-couplings of the third order solutions. In order to derive Eqs. (46), (47) and (48), we define

$$\mathcal{W}_{\gamma\alpha}(\mathbf{p}) \equiv \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\gamma\alpha(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3), \tag{B1}$$

$$\mathcal{W}_{\gamma\gamma}(\mathbf{p}) \equiv \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\gamma\gamma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3), \tag{B2}$$

$$\mathcal{W}_{\xi}(\mathbf{p}) \equiv \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\xi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3), \tag{B3}$$

with

$$\gamma\alpha(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{3}\Big(\gamma(\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3)\alpha^{(s)}(\mathbf{k}_2,\mathbf{k}_3) + 2\text{ cyclic terms}\Big), \tag{B4}$$

$$\gamma\gamma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{3}\left(\gamma(\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3)\gamma(\mathbf{k}_2,\mathbf{k}_3) + 2\text{ cyclic terms}\right), \tag{B5}$$

$$\xi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = 1 - \frac{k_1^2(\mathbf{k}_2\cdot\mathbf{k}_3)^2 + k_2^2(\mathbf{k}_3\cdot\mathbf{k}_1)^2 + k_3^2(\mathbf{k}_1\cdot\mathbf{k}_2)^2 - 2(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{k}_3)(\mathbf{k}_3\cdot\mathbf{k}_1)}{k_1^2k_2^2k_3^2}. \tag{B6}$$

In deriving Eqs. (54) and (55), we define

$$\mathcal{W}_{\alpha\alpha R}(\mathbf{p}) = \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\alpha\alpha_R(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3), \quad \text{(B7)}$$

$$\mathcal{W}_{\alpha\gamma R}(\mathbf{p}) = \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\alpha\gamma_R(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3), \quad \text{(B8)}$$

$$\mathcal{W}_{\alpha\alpha L}(\mathbf{p}) = \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\alpha\alpha_L(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3), \quad \text{(B9)}$$

$$\mathcal{W}_{\alpha\gamma L}(\mathbf{p}) = \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\alpha\gamma_L(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3), \quad \text{(B10)}$$

$$\mathcal{W}_{\alpha\alpha}(\mathbf{p}) = \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\alpha\alpha(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3), \quad \text{(B11)}$$

$$\mathcal{W}_{\alpha\gamma}(\mathbf{p}) = \frac{1}{(2\pi)^6}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{p})\alpha\gamma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_L(\mathbf{k}_1)\delta_L(\mathbf{k}_2)\delta_L(\mathbf{k}_3) \quad \text{(B12)}$$

with

$$\alpha\alpha_R(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{3}\left(\alpha(\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3)\alpha^{(s)}(\mathbf{k}_2,\mathbf{k}_3) + 2 \text{ cyclic terms}\right), \quad \text{(B13)}$$

$$\alpha\gamma_R(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{3}\left(\alpha(\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3)\gamma(\mathbf{k}_2,\mathbf{k}_3) + 2 \text{ cyclic terms}\right), \quad \text{(B14)}$$

$$\alpha\alpha_L(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{3}\left(\alpha(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3)\alpha^{(s)}(\mathbf{k}_2,\mathbf{k}_3) + 2 \text{ cyclic terms}\right), \quad \text{(B15)}$$

$$\alpha\gamma_L(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{3}\left(\alpha(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3)\gamma(\mathbf{k}_2,\mathbf{k}_3) + 2 \text{ cyclic terms}\right), \quad \text{(B16)}$$

$$\alpha\alpha(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{3}\left(\alpha^{(s)}(\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3)\alpha^{(s)}(\mathbf{k}_2,\mathbf{k}_3) + 2 \text{ cyclic terms}\right), \quad \text{(B17)}$$

$$\alpha\gamma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{3}\left(\alpha^{(s)}(\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3)\gamma(\mathbf{k}_2,\mathbf{k}_3) + 2 \text{ cyclic terms}\right). \quad \text{(B18)}$$

Appendix C: Derivation of the 1-loop power spectra

The cosmological density contrast $\delta(t,\mathbf{k})$ and the velocity divergence $\theta(t,\mathbf{k})$ up to the third order of the perturbative expansion are expressed as

$$\delta(t,\mathbf{k}) = D_+(t)\delta_{\mathrm{L}}(\mathbf{k}) + D_+^2(t)\delta_{2K}(t,\mathbf{k}) + D_+^3(t)\delta_{3K}(t,\mathbf{k}), \quad \text{(C1)}$$

$$\theta(t,\mathbf{k}) = -f\left(D_+(t)\delta_{\mathrm{L}}(\mathbf{k}) + D_+^2(t)\theta_{2K}(t,\mathbf{k}) + D_+^3(t)\theta_{3K}(t,\mathbf{k})\right), \quad \text{(C2)}$$

were we define

$$\delta_{2K}(t,\mathbf{k}) = \mathcal{W}_\alpha(\mathbf{k}) - \frac{2}{7}\lambda(t)\mathcal{W}_\gamma(\mathbf{k}), \quad \text{(C3)}$$

$$\delta_{3K}(t,\mathbf{k}) = \mathcal{W}_{\alpha\alpha}(\mathbf{k}) - \frac{2}{7}\lambda(t)\mathcal{W}_{\alpha\gamma R}(\mathbf{k}) - \frac{2}{7}\lambda(t)\mathcal{W}_{\alpha\gamma L}(\mathbf{k}) - \frac{2}{21}\mu(t)\mathcal{W}_{\gamma\gamma}(\mathbf{k}) + \frac{1}{9}\nu(t)\mathcal{W}_\xi(\mathbf{k}), \quad \text{(C4)}$$

$$\theta_{2K}(t,\mathbf{k}) = \mathcal{W}_\alpha(\mathbf{k}) - \frac{4}{7}\lambda_\theta(t)\mathcal{W}_\gamma(\mathbf{k}), \quad \text{(C5)}$$

$$\theta_{3K}(t,\mathbf{k}) = \mathcal{W}_{\alpha\alpha}(\mathbf{k}) - \frac{4}{7}\lambda_\theta(t)\mathcal{W}_{\alpha\gamma R}(\mathbf{k}) - \frac{2}{7}\lambda(t)\mathcal{W}_{\alpha\gamma L}(\mathbf{k}) - \frac{2}{7}\mu_\theta(t)\mathcal{W}_{\gamma\gamma}(\mathbf{k}) + \frac{1}{3}\nu_\theta(t)\mathcal{W}_\xi(\mathbf{k}), \quad \text{(C6)}$$

and the kernels for the density contrast F_2, and F_3, and those for the velocity divergence G_2, and G_3, as follows,

$$F_2(t,\mathbf{k}_1,\mathbf{k}_2) = \alpha(\mathbf{k}_1,\mathbf{k}_2) - \frac{2}{7}\lambda(t)\gamma(\mathbf{k}_1,\mathbf{k}_2), \tag{C7}$$

$$G_2(t,\mathbf{k}_1,\mathbf{k}_2) = \alpha(\mathbf{k}_1,\mathbf{k}_2) - \frac{4}{7}\lambda_\theta(t)\gamma(\mathbf{k}_1,\mathbf{k}_2), \tag{C8}$$

$$\begin{aligned}F_3(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= \alpha\alpha(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - \frac{2}{7}\lambda(t)\alpha\gamma_R(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - \frac{2}{7}\lambda(t)\alpha\gamma_L(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\\ &\quad -\frac{2}{21}\mu(t)\gamma\gamma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + \frac{1}{9}\nu(t)\xi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3),\end{aligned} \tag{C9}$$

$$\begin{aligned}G_3(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= \alpha\alpha(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - \frac{4}{7}\lambda_\theta(t)\alpha\gamma_R(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - \frac{2}{7}\lambda(t)\alpha\gamma_L(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\\ &\quad -\frac{2}{7}\mu_\theta(t)\gamma\gamma(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + \frac{1}{3}\nu_\theta(t)\xi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3).\end{aligned} \tag{C10}$$

These kernels have the two types of symmetries. One is the symmetries in replacement of the wave numbers,

$$F_2(t,\mathbf{k}_1,\mathbf{k}_2) = F_2(t,\mathbf{k}_2,\mathbf{k}_1), \tag{C11}$$

$$\begin{aligned}F_3(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= F_3(t,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_1) = F_3(t,\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_2)\\ &= F_3(t,\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2) = F_3(t,\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_3) = F_3(t,\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_1).\end{aligned} \tag{C12}$$

The second is the symmetries in the conversion of the sign of the wavenumbers,

$$F_2(t,\mathbf{k}_1,\mathbf{k}_2) \;=\; F_2(t,-\mathbf{k}_1,-\mathbf{k}_2), \tag{C13}$$

$$F_3(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \;=\; F_3(t,-\mathbf{k}_1,-\mathbf{k}_2,-\mathbf{k}_3). \tag{C14}$$

The same relations hold for $G_2(t,\mathbf{k}_1,\mathbf{k}_2)$ and $G_3(t,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$.

The above properties are useful in deriving the expressions of the power spectra, $P_{\delta\delta}(t,k)$, $P_{\delta\theta}(t,k)$, $P_{\theta\theta}(t,k)$, defined by Eqs. (85), (86) and (87). Using the expressions (C1) and (C2), we find

$$P_{\delta\delta}(t,k) \;=\; D_+^2(t)P_{\rm L}(k) + D_+^4(t)\left(P_{\delta\delta}^{(22)}(t,k) + 2P_{\delta\delta}^{(13)}(t,k)\right), \tag{C15}$$

$$P_{\delta\theta}(t,k) \;=\; D_+^2(t)P_{\rm L}(k) + D_+^4(t)\left(P_{\delta\theta}^{(22)}(t,k) + 2P_{\delta\theta}^{(13)}(t,k)\right), \tag{C16}$$

$$P_{\theta\theta}(t,k) \;=\; D_+^2(t)P_{\rm L}(k) + D_+^4(t)\left(P_{\theta\theta}^{(22)}(t,k) + 2P_{\theta\theta}^{(13)}(t,k)\right), \tag{C17}$$

where $D_+^2(t)P_{\rm L}(k)$ is the linear matter power spectrum,

$$\langle\delta_{\rm L}(\mathbf{k}_1)\delta_{\rm L}(\mathbf{k}_2)\rangle \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_{\rm L}(k), \tag{C18}$$

and we define

$$\langle\delta_{2K}(t,\mathbf{k}_1)\delta_{2K}(t,\mathbf{k}_2)\rangle \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_{\delta\delta}^{(22)}(t,k), \tag{C19}$$

$$\langle\delta_{\rm L}(\mathbf{k}_1)\delta_{3K}(t,\mathbf{k}_2)\rangle \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_{\delta\delta}^{(13)}(t,k), \tag{C20}$$

$$\langle\delta_{2K}(t,\mathbf{k}_1)\theta_{2K}(t,\mathbf{k}_2)\rangle \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_{\delta\theta}^{(22)}(t,k), \tag{C21}$$

$$\langle\theta_{2K}(t,\mathbf{k}_1)\theta_{2K}(t,\mathbf{k}_2)\rangle \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_{\theta\theta}^{(22)}(t,k), \tag{C22}$$

$$\langle\delta_{\rm L}(\mathbf{k}_1)\theta_{3K}(t,\mathbf{k}_2)\rangle \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_{\theta\theta}^{(13)}(t,k), \tag{C23}$$

and

$$\frac{1}{2}\left(\langle\delta_{\rm L}(\mathbf{k}_1)\theta_{3K}(t,\mathbf{k}_2)\rangle + \langle\delta_{3K}(t,\mathbf{k}_1)\delta_{\rm L}(\mathbf{k}_2)\rangle\right) \;=\; (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_{\delta\theta}^{(13)}(t,k). \tag{C24}$$

As an example, let us explain the derivation of $P_{\delta\delta}^{(22)}(t,k)$. Inserting (C3) into (C19), we have

$$\begin{aligned}\langle\delta_{2K}(t,\mathbf{k}_1)\delta_{2K}(t,\mathbf{k}_2)\rangle &= \Big\langle\frac{1}{(2\pi)^3}\int d^3q_1d^3q_2\delta^{(3)}(\mathbf{k}_1-\mathbf{q}_1-\mathbf{q}_2)F_2(t,\mathbf{q}_1,\mathbf{q}_2)\delta_{\rm L}(\mathbf{q}_1)\delta_{\rm L}(\mathbf{q}_2)\\ &\qquad\times\frac{1}{(2\pi)^3}\int d^3q_3d^3q_4\delta^{(3)}(\mathbf{k}_2-\mathbf{q}_3-\mathbf{q}_4)F_2(t,\mathbf{q}_3,\mathbf{q}_4)\delta_{\rm L}(\mathbf{q}_3)\delta_{\rm L}(\mathbf{q}_4)\Big\rangle\\ &= \frac{1}{(2\pi)^6}\int d^3q_1d^3q_3F_2(t,\mathbf{q}_1,\mathbf{k}_1-\mathbf{q}_1)F_2(t,\mathbf{q}_3,\mathbf{k}_2-\mathbf{q}_3)\\ &\qquad\times\langle\delta_L(\mathbf{q}_1)\delta_L(\mathbf{k}_1-\mathbf{q}_1)\delta_L(\mathbf{q}_3)\delta_L(\mathbf{k}_2-\mathbf{q}_3)\rangle\,.\end{aligned}\tag{C25}$$

Using the relation that hold for the Gaussian variables, we have

$$\begin{aligned}\langle\delta_L(\mathbf{q}_1)\delta_L(\mathbf{k}_1-\mathbf{q}_1)\delta_L(\mathbf{q}_3)\delta_L(\mathbf{k}_2-\mathbf{q}_3)\rangle &= \langle\delta_L(\mathbf{q}_1)\delta_L(\mathbf{k}_1-\mathbf{q}_1)\rangle\langle\delta_L(\mathbf{q}_3)\delta_L(\mathbf{k}_2-\mathbf{q}_3)\rangle\\ &+\langle\delta_L(\mathbf{q}_1)\delta_L(\mathbf{q}_3)\rangle\langle\delta_L(\mathbf{k}_2-\mathbf{q}_3)\delta_L(\mathbf{k}_1-\mathbf{q}_1)\rangle\\ &+\langle\delta_L(\mathbf{q}_1)\delta_L(\mathbf{k}_2-\mathbf{q}_3)\rangle\langle\delta_L(\mathbf{k}_1-\mathbf{q}_1)\delta_L(\mathbf{q}_3)\rangle\,,\end{aligned}\tag{C26}$$

which yields

$$\begin{aligned}&\langle\delta_L(\mathbf{k}_{11})\delta_L(\mathbf{k}_1-\mathbf{k}_{11})\delta_L(\mathbf{k}_{21})\delta_L(\mathbf{k}_2-\mathbf{k}_{21})\rangle\\ &\qquad=(2\pi)^6\delta^{(3)}(\mathbf{q}_1+\mathbf{q}_3)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{q}_1-\mathbf{q}_3)P_{\rm L}(q_1)P_{\rm L}(|\mathbf{k}_1-\mathbf{q}_1|)\\ &\qquad+(2\pi)^6\delta^{(3)}(\mathbf{q}_1+\mathbf{k}_2-\mathbf{q}_3)\delta^{(3)}(\mathbf{k}_1-\mathbf{q}_1+\mathbf{q}_3)P_{\rm L}(q_1)P_{\rm L}(q_3),\end{aligned}\tag{C27}$$

with Eq. (C18). Then, (C25) yields

$$\begin{aligned}\langle\delta_{2K}(t,\mathbf{k}_1)\delta_{2K}(t,\mathbf{k}_2)\rangle &= \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\int d^3q_1\Big(F_2(t,\mathbf{q}_1,\mathbf{k}_1-\mathbf{q}_1)F_2(t,-\mathbf{q}_1,\mathbf{k}_2+\mathbf{q}_1)\\ &\qquad+F_2(t,\mathbf{q}_1,\mathbf{k}_1-\mathbf{q}_1)F_2(t,\mathbf{k}_2+\mathbf{q}_1,-\mathbf{q}_1)\Big)P_{\rm L}(q_1)P_{\rm L}(|\mathbf{k}_1-\mathbf{q}_1|).\end{aligned}\tag{C28}$$

Using the relation (C13), we have

$$P_{\delta\delta}^{(22)}(t,k)=\frac{2}{(2\pi)^3}\int d^3q_1F_2^2(t,\mathbf{q}_1,\mathbf{k}-\mathbf{q}_1)P_{\rm L}(q_1)P_{\rm L}(|\mathbf{k}-\mathbf{q}_1|),\tag{C29}$$

which reduces to (91). In the derivation, we define $x=\cos\theta$, where θ is the angle between $\mathbf{k}_1$ and $\mathbf{q}_1$. Similarly, the expressions (92) and (93) are obtained for $P_{\delta\theta}^{(22)}(t,k)$ and $P_{\theta\theta}^{(22)}(t,k)$. In the limit of the Einstein de Sitter universe withtin the general relativity, $\lambda(t)=\lambda_\theta(t)=\mu(t)=\mu_\theta(t)=1$, which gives the well-known expressions

$$P_{\delta\delta}^{(22)}(k) = \frac{k^3}{98(2\pi)^2}\int drP_{\rm L}(rk)\int_{-1}^1 dxP_{\rm L}((1+r^2-2rx)^{1/2})\frac{(3r+7x-10rx^2)^2}{(1+r^2-2rx)^2},\tag{C30}$$

$$\begin{aligned}P_{\delta\theta}^{(22)}(k) &= \frac{k^3}{98(2\pi)^2}\int drP_{\rm L}(rk)\int_{-1}^1 dxP_{\rm L}((1+r^2-2rx)^{1/2})\\ &\qquad\times\frac{(3r+7x-10rx^2)(-r+7x-6rx^2)}{(1+r^2-2rx)^2}\end{aligned}\tag{C31}$$

$$P_{\theta\theta}^{(22)}(k) = \frac{k^3}{98(2\pi)^2}\int drP_{\rm L}(rk)\int_{-1}^1 dxP_{\rm L}((1+r^2-2rx)^{1/2})\frac{(-r+7x-6rx^2)^2}{(1+r^2-2rx)^2},\tag{C32}$$

which are constant as functions of time.

Next, let us explain the derivation of $P_{\delta\delta}^{(13)}(t,k)$. Inserting (C4) into (C20), we have

$$\begin{aligned}&2\langle\delta_L(\mathbf{k}_1)\delta_{3K}(t,\mathbf{k}_2)\rangle\\ &\quad=\frac{2}{(2\pi)^6}\int d^3q_1d^3q_2F_3(t,\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}_2-\mathbf{q}_1-\mathbf{q}_2)\langle\delta_L(\mathbf{k}_1)\delta_L(\mathbf{q}_1)\delta_L(\mathbf{q}_2)\delta_L(\mathbf{k}_2-\mathbf{q}_1-\mathbf{q}_2)\rangle\,.\end{aligned}\tag{C33}$$

Using the relations,

$$\begin{aligned}\langle\delta_L(\mathbf{k}_1)\delta_L(\mathbf{q}_1)\delta_L(\mathbf{q}_2)\delta_L(\mathbf{k}_2-\mathbf{q}_1-\mathbf{q}_2)\rangle &= (2\pi)^6\delta^{(3)}(\mathbf{k}_1+\mathbf{q}_1)P_{\rm L}(k_1)\delta^{(3)}(\mathbf{k}_2-\mathbf{q}_1)P_{\rm L}(q_2)\\ &+ (2\pi)^6\delta^{(3)}(\mathbf{k}_1+\mathbf{q}_2)P_{\rm L}(k_1)\delta^{(3)}(\mathbf{k}_2-\mathbf{q}_2)P_{\rm L}(q_1)\\ &+ (2\pi)^6\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{q}_1-\mathbf{q}_2)P_{\rm L}(\mathbf{k}_1)\delta^{(3)}(\mathbf{q}_1+\mathbf{q}_2)P_{\rm L}(q_1),\end{aligned} \tag{C34}$$

and the symmetries, (C12), we have

$$2P^{(13)}_{\delta\delta}(t,k)=\frac{6}{(2\pi)^3}\int d^3q_1F_3(t,\mathbf{k},\mathbf{q}_1,-\mathbf{q}_1)P_{\rm L}(k)P_{\rm L}(q_1). \tag{C35}$$

After performing the angular integration with respect to the spherical coordinate of $\mathbf{q_1}$, we finally have (94). Note that (94) does not depend on $\nu(t)$, which occurs because of the identity $\xi(\mathbf{k},\mathbf{q}_1,-\mathbf{q}_1)=0$. $P^{(13)}_{\delta\delta}(t,k)$ is characterized by $\lambda(t)$ and $\mu(t)$. Similarly, we have the expressions (95) and (96) for $P^{(13)}_{\delta\theta}(t,k)$ and $P^{(13)}_{\theta\theta}(t,k)$, respectively. Because of the same reason for $P^{(13)}_{\delta\delta}(t,k)$, $P^{(13)}_{\delta\theta}(t,k)$ and $P^{(13)}_{\theta\theta}(t,k)$ do not depend on $\nu(t)$ and $\nu_\theta(t)$. Furthermore, $P^{(13)}_{\delta\theta}(t,k)$ and $P^{(13)}_{\theta\theta}(t,k)$ do not depend on $\lambda_\theta(t)$. This is because of the nature of the integration

$$\int dx\ \alpha\gamma_R(\mathbf{k},\mathbf{q}_1,-\mathbf{q}_1)=0. \tag{C36}$$

Finally, $P^{(13)}_{\delta\theta}(t,k)$ depends on $\lambda(t)$, $\mu(t)$, and $\mu_\theta(t)$, and $P^{(13)}_{\theta\theta}(t,k)$ depends on $\lambda(t)$ and $\mu_\theta(t)$. We find the following relation holds, in general, $P^{(13)}_{\delta\theta}(t,k)=[P^{(13)}_{\delta\delta}(t,k)+P^{(13)}_{\theta\theta}(t,k)]/2$, from (C24).

In the limit of the Einstein de Sitter universe withtin the general relativity, all the coefficients $\lambda(t)$, $\mu(t)$, $\mu_\theta(t)$ reduce to 1, which reproduces the well-known expressions

$$\begin{aligned}2P^{(13)}_{\delta\delta}(k) &= \frac{k^3}{252(2\pi)^2}P_{\rm L}(k)\int drP_{\rm L}(rk)\\ &\times\left[12\frac{1}{r^2}-158+100r^2-42r^4+\frac{3}{r^3}(r^2-1)^3(7r^2+2)\ln\left(\frac{r+1}{|r-1|}\right)\right],\end{aligned} \tag{C37}$$

$$\begin{aligned}2P^{(13)}_{\delta\theta}(k) &= \frac{k^3}{252(2\pi)^2}P_{\rm L}(k)\int drP_{\rm L}(rk)\\ &\times\left[24\frac{1}{r^2}-202+56r^2-30r^4+\frac{3}{r^3}\left(r^2-1\right)^3\left(5r^2+4\right)\ln\left(\frac{r+1}{|r-1|}\right)\right],\end{aligned} \tag{C38}$$

$$\begin{aligned}2P^{(13)}_{\theta\theta}(k) &= \frac{k^3}{84(2\pi)^2}P_{\rm L}(k)\int drP_{\rm L}(rk)\\ &\times\left[12\frac{1}{r^2}-82+4r^2-6r^4+\frac{3}{r^3}\left(r^2-1\right)^3\left(r^2+2\right)\ln\left(\frac{r+1}{|r-1|}\right)\right].\end{aligned} \tag{C39}$$

Appendix D: The integrations of mode-coupling functions

Here we summarize the useful expressions, which are useful in deriving the 1-loop order power spectra,

$$\alpha^2(\mathbf{q}_1,\mathbf{k}_1-\mathbf{q}_1) = \frac{(r+x-2rx^2)^2}{4r^2(1+r^2-2rx)^2}, \tag{D1}$$

$$\alpha(\mathbf{q}_1,\mathbf{k}_1-\mathbf{q}_1)\gamma(\mathbf{q}_1,\mathbf{k}_1-\mathbf{q}_1) = \frac{(r+x-2rx^2)(-1+x^2)}{2r(1+r^2-2rx)^2}, \tag{D2}$$

$$\gamma^2(\mathbf{q}_1,\mathbf{k}_1-\mathbf{q}_1) = \frac{(-1+x^2)^2}{(1+r^2-2rx)^2}, \tag{D3}$$

and

$$\int d^3q_1\ P_{\rm L}(rk)\alpha\alpha(\mathbf{k},\mathbf{q}_1,-\mathbf{q}_1) = \frac{2\pi k^3}{72}\int drP_{\rm L}(rk)\left[-2+16r^2-6r^4+\frac{3}{r^3}(r^2-1)^3\ln\left(\frac{r+1}{|r-1|}\right)\right], \tag{D4}$$

$$\int d^3q_1\ P_{\rm L}(rk)\alpha\gamma_R(\mathbf{k},\mathbf{q}_1,-\mathbf{q}_1) = 0, \tag{D5}$$

$$\int d^3q_1\ P_{\rm L}(rk)\alpha\gamma_L(\mathbf{k},\mathbf{q}_1,-\mathbf{q}_1) = \frac{2\pi k_1^3}{36}\int drP_{\rm L}(rk)\left[6+16r^2-6r^4+\frac{3}{r^3}(r^2-1)^3\ln\left(\frac{r+1}{|r-1|}\right)\right], \tag{D6}$$

$$\int d^3q_1\ P_{\rm L}(rk)\gamma\gamma(\mathbf{k}_1,\mathbf{q}_1,-\mathbf{q}_1) = \frac{2\pi k_1^3}{72}\int drP_{\rm L}(rk)\left[-6\frac{1}{r^2}+22+22r^2-6r^4+\frac{3}{r^3}(r^2-1)^4\ln\left(\frac{r+1}{|r-1|}\right)\right], \tag{D7}$$

$$\int d^3q_1\ P_{\rm L}(rk)\xi(\mathbf{k}_1,\mathbf{q}_1,-\mathbf{q}_1) = 0. \tag{D8}$$

Appendix E: Coefficients and in the KGB model

In the KGB model, we find the coefficients in basic equations,

$$\mathcal{F}_T = M_{\rm pl}^2, \qquad \mathcal{G}_T = M_{\rm pl}^2, \tag{E1}$$

$$\Theta = -nM_{\rm pl}\left(\frac{r_c^2}{M_{\rm pl}^2}\right)^n\dot\phi X^n + HM_{\rm pl}^2, \tag{E2}$$

$$\dot\Theta = -n(2n+1)M_{\rm pl}\left(\frac{r_c^2}{M_{\rm pl}^2}\right)^n\ddot\phi X^n + \dot HM_{\rm pl}^2, \tag{E3}$$

$$\mathcal{E} = -X + 6nM_{\rm pl}\left(\frac{r_c^2}{M_{\rm pl}^2}\right)^n\dot\phi HX^n - 3H^2M_{\rm pl}^2, \tag{E4}$$

$$\mathcal{P} = -X - 2nM_{\rm pl}\left(\frac{r_c^2}{M_{\rm pl}^2}\right)^n\ddot\phi X^n + (3H^2+2\dot H)M_{\rm pl}^2, \tag{E5}$$

$$A_0 = \frac{X}{H^2} - 2nM_{\rm pl}\left(\frac{r_c^2}{M_{\rm pl}^2}\right)^n\left(\frac{2\dot\phi}{H}+n\frac{\ddot\phi}{H^2}\right)X^n, \tag{E6}$$

$$A_2 = B_0 = n\frac{\dot\phi}{H}M_{\rm pl}\left(\frac{r_c^2}{M_{\rm pl}^2}\right)^nX^n, \tag{E7}$$

$$A_1 = B_1 = B_2 = B_3 = C_0 = C_1 = 0, \tag{E8}$$

and the non-trivial expressions,

$$L(t) = -\frac{A_0\mathcal{F}_T\rho_{\rm m}}{2(A_0\mathcal{G}_T + A_2^2\mathcal{F}_T)}, \tag{E9}$$

$$N_\gamma(t) = \frac{B_0A_2^3\mathcal{F}_T^3\rho_{\rm m}^2}{4(A_0\mathcal{G}_T^2 + A_2^2\mathcal{F}_T)^3H^2}, \tag{E10}$$

$$H^2\mu_\Phi = -\frac{8B_0\mathcal{T}^3\rho_{\rm m}^2}{7H^2\mathcal{Z}^3}\lambda - \frac{8B_0^2\mathcal{G}_T^2\mathcal{T}^4\rho_{\rm m}^3}{H^4\mathcal{Z}^5}. \tag{E11}$$

We use the attractor solution which satisfies $3\dot{\phi}HG_{3X}=1$. Then we have

$$\ddot{\phi} = -\frac{1}{2n-1}\frac{\dot{\phi}\dot{H}}{H}, \tag{E12}$$

$$\frac{\dot{H}}{H^2} = -\frac{(2n-1)3\Omega_{\rm m}}{2(2n-\Omega_{\rm m})}, \tag{E13}$$

$$A_0 = -\frac{M_{\rm pl}^2(1-\Omega_{\rm m})\left(2n+(3n-1)\Omega_{\rm m}\right)}{2n-\Omega_{\rm m}}, \tag{E14}$$

$$A_2 = M_{\rm pl}^2(1-\Omega_{\rm m}), \tag{E15}$$

$$B_0 = M_{\rm pl}^2(1-\Omega_{\rm m}), \tag{E16}$$

where we define $\Omega_{\rm m}=\rho_m(a)/3M_{\rm pl}^2H^2$. We also have

$$\mathcal{R} = -\frac{M_{\rm pl}^4(1-\Omega_{\rm m})\left(2n+(3n-1)\Omega_{\rm m}\right)}{2n-\Omega_{\rm m}}, \tag{E17}$$

$$\mathcal{S} = -\frac{M_{\rm pl}^4(1-\Omega_{\rm m})\left(2n+(3n-1)\Omega_{\rm m}\right)}{2n-\Omega_{\rm m}}, \tag{E18}$$

$$\mathcal{T} = M_{\rm pl}^4(1-\Omega_{\rm m}), \tag{E19}$$

$$\mathcal{Z} = 2\frac{M_{\rm pl}^6\Omega_{\rm m}(5n-\Omega_{\rm m})(1-\Omega_{\rm m})}{2n-\Omega_{\rm m}}. \tag{E20}$$

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